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ELEMENTARY ALGEBRA

SECOND COURSE

BY

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PREFACE

THIS volume has been prepared as a companion to the FIRST COURSE in elementary algebra, written by the same authors. As its title suggests, the SECOND COURSE is designed for students who have had the equivalent of the work given in the FIRST COURSE. The FIRST COURSE gives one year's work, including the topics now generally taught in the first year of the average high school. The SECOND COURSE reviews the topics treated in the FIRST COURSE, amplifies the treatment in the latter by the addition of new matter where necessary, and extends the course to include all topics required for admission to college.

In the preparation of this book, the authors have been guided by the same aims and educational principles as in the preparation of the FIRST COURSE. The distinguishing features of the latter will be found also in the present volume.

The book contains a maximum of practice, abundance of drill being provided throughout. New principles and processes are adequately and psychologically developed, *only one new difficulty being presented at a time*, and these are finally mastered through plentiful application in drill exercises and problems.

A fundamental aim here, as in the other books by the same authors, has been to lead the student to see the subject, not merely as a system of exercises to be pursued for the purpose of mental discipline, although the authors are thorough believers in this feature of study, but rather as a scientific instrument for solving certain types of problems actually encountered in the world's work. To this end adequate use has been made of real applied problems of the various types encountered in practical life. These real applied problems lend intense interest, and motivate the work legitimately. Furthermore, it is through the application of the student's knowledge of algebra in the solution of these real applied problems that the knowledge is led to function. The old-time useless puzzles have been eliminated from the book.

Algebra has been correlated with arithmetic and geometry. The principles of algebra have been applied to shorten arithmetical computations. See, for example, § 3, § 5, § 11, exercises of § 15, etc. Numerical problems have been given in many lists of exercises. See, for example, exercises pp. 16, 34, 35, 39, 40, 41, 43, etc. Considerable use of percentage has been made in the solution of certain types of problems in business, etc. The facts of mensuration have been much used, and some knowledge of simple facts in geometry with which the student is familiar has been drawn upon in problems.

Graphical methods have been used in a natural way throughout the book, both in the interpretation and illustration of algebraic principles and in the solution of certain types of problems. See, for example, §§ 3, 11, 32, 51, 53, 68, 84, 85, 95, 98, 100, 109, etc.

Supplementary exercises have been given at the end of each chapter. They adapt the text to the use of schools with different kinds of courses. These exercises, as a rule, are more difficult than those in the body of the chapters. They may be used for reviews or in classes wanting a more difficult course; or they may be omitted entirely without interfering with the subsequent work.

The authors wish to acknowledge their indebtedness to all those whose timely suggestions and criticisms have helped in the preparation of this textbook.

JOHN C. STONE.
JAMES F. MILLIS.

SEPTEMBER, 1912.

CONTENTS

CHAPTER PAGE

I. THE FUNDAMENTAL OPERATIONS 1

II. FACTORS AND MULTIPLES 22

III. FRACTIONS 37

IV. LINEAR EQUATIONS IN ONE UNKNOWN NUMBER . . 49

V. SYSTEMS OF LINEAR EQUATIONS 64

VI. ROOTS, SURDS, AND IMAGINARY NUMBERS . . . 85

VII. QUADRATIC EQUATIONS 112

VIII. HIGHER EQUATIONS 131

IX. EQUATIONS INVOLVING SURDS 13

X. SYSTEMS INVOLVING QUADRATIC AND HIGHER EQUATIONS 143

XI. PROPORTION. VARIABLES 167

XII. PROGRESSIONS 190

XIII. EXPONENTS 211

XIV. THE BINOMIAL THEOREM 222

XV. LOGARITHMS 230

MISCELLANEOUS EXERCISES 245

ELEMENTARY ALGEBRA

SECOND COURSE

CHAPTER I

THE FUNDAMENTAL OPERATIONS

1. The principles and processes of algebra established in the FIRST COURSE will be employed throughout the more advanced work. Some of the most important rules that were carefully established in the FIRST COURSE will be restated in the following sections. Also, in connection with these rules, there will be introduced additional matter that was omitted from the FIRST COURSE, either because it was not needed, or because it was too difficult for a first reading.

2. **Number Expressions.** — In algebra and in other branches of mathematics, letters are used to represent numbers with *unknown* or *unassigned* values.

Thus, in πr^2, which expresses the area of a circle, r represents the length of the radius of any circle, which may have any numerical value, depending upon the size of the circle. Similarly, in $a - b$ the letters a and b represent numbers to which may be assigned any numerical values whatsoever.

A number represented by a letter is called a **literal number**, and any number expression in which one or more number symbols are letters is called a **literal expression**.

In any number expression the parts connected by the signs + or − are called **terms**. An expression not connected to any other expression by either of the signs + or − consists of *one term*, and is called a **monomial**. An expression consisting of *two terms* is a **binomial**. An expression consisting of *three terms* is a **trinomial**.

In general, an expression consisting of *two* or *more terms* is called a **polynomial.**

If numerical values are assigned to all of the literal numbers in any literal expression, a particular numerical value of the whole expression may be found by performing the operations indicated in it.

In finding the value of an expression containing more than one term, the value of each term must be found before the additions or subtractions indicated between the terms.

EXERCISES

1. How is the product of a numerical and a literal factor, or of two or more literal factors, indicated?

2. Express the product of 5 and n; of x and y; of 2, a, and b.

3. How is the product of two polynomials indicated?

4. Indicate the product of $t-4$ and $2t+3$.

5. What is the meaning of a^2? Of n^3? Of $(x-y)^4$?

6. In m^5, what name is given to 5? To m? To the whole expression?

7. In $8x$, what is the 8 called?

8. Name the numerical coefficients in $2A$, $12x^2$, and $6mn$.

9. Do $3a$ and a^3 mean the same? What are their values when a is 4? When $a=10$?

10. Tell how many terms in each of the following, and apply to each the appropriate name, "monomial," etc.: $4\pi R^2$; a^2-b^2; $2x-y+7z$; B^2-4AC; $a^2+2ab+b^2-c^2$; $.05V^2$; $(m+n)(m-n)$; W^2g-2Wg^2; $x^3-3x^2y+3xy^2-y^3$; $a^4+a^3+a^2+a+1$.

11. Find the value of N^4 when $N=3$; when $N=4$.

12. Find the value of $5x^2y$ when $x=2$ and $y=3$.

13. Find the value of a^2-b^2 when $a=5$ and $b=3$.

14. Find the value of $A^2-2AB+B^2$ when $A=8$ and $B=1$.

15. Find the value of $2w^3+v^3$ when $w=3$ and $v=5$.

16. Find the value of $(2x+y)(4x-3y)$ when $x=9$ and $y=4$.

17. Find the value of $(R-at)^2$ when $R=16$, $a=32$, and $t=\frac{1}{4}$.

18. The volume of a sphere whose radius is R is $\frac{4}{3}\pi R^3$. Find the volume when $R=3$ in. (Use $\pi=3.1416$.)

19. The area of a trapezoid whose altitude is h and bases b and and b', respectively, is $\frac{1}{2}h(b+b')$. Find the area when $h=12$ in., $b=7$ in., and $b'=15$ in.

20. The cubic contents or volume of any frustum of a circular cone, such as a coffeepot, or wash tub, whose depth is H, radius at the bottom R, and radius at the top r, is $\frac{1}{3}\pi H(R^2+Rr+r^2)$. Find the cubic contents of a tub whose depth is 12 in., radius at the bottom 12 in., and radius at the top 16 in.

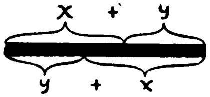
3. Fundamental Laws of Addition. — In the addition of two or more numbers, it is evident that the value of the sum is the same in whatever order the numbers are added. This truth is called the **Law of Order in Addition.**

In the case of two numbers the law may be expressed by

$$x+y=y+x.$$

It may be shown graphically as in the margin.

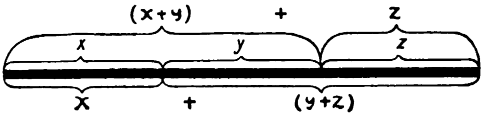
It is evident also that if three or more numbers are to be added, they may be combined or grouped in any way. In $a+b+c+d$, instead of first adding b to a, then c to the sum, etc., we may first add b to a, then d to c, then add the sums. Or we may first add c to b, then d to the sum, then this sum to a, etc. This truth is called the **Law of Grouping in Addition.**



In the case of three numbers the law may be expressed by

$$(x+y)+z=x+(y+z).$$

It may be illustrated graphically as follows:



These two laws may be used to shorten arithmetical addition.

Thus, $6+5+4+3+2$ may be written $6+4+5+3+2$, by the Law of Order, and then $(6+4)+(5+3+2)$, by the Law of Grouping. Adding 6 and 4 gives 10, adding 5, 3, and 2 gives 10, and adding these sums gives 20. In practice, the numbers are written in a column, as in the margin. The change of order is made mentally, and the numbers that make 10 are grouped.

$$\begin{array}{r}6\\5\\4\\3\\\underline{2}\\20\end{array}$$

EXERCISES

Add by grouping numbers that make 10:

1. 7, 8, 3, 2.
2. 1, 6, 9, 4.
3. 3, 5, 4, 6, 1.
4. 5, 9, 2, 1, 3.
5. 8, 6, 2, 7, 4, 3.
6. 4, 7, 7, 6, 9, 3.

Add by grouping numbers in each column:

$$7.\ \begin{array}{r}492\\865\\628\\\underline{315}\end{array}\qquad 8.\ \begin{array}{r}347\\951\\763\\\underline{426}\end{array}\qquad 9.\ \begin{array}{r}827\\584\\276\\\underline{405}\end{array}\qquad 10.\ \begin{array}{r}3526\\5982\\7192\\\underline{8304}\end{array}$$

11. Show graphically that $a+b+c+d=b+a+d+c$.

12. Show graphically that $a+b+c+d=a+(b+c+d)$.

4. Addition of Positive and Negative Numbers. — For the addition of positive and negative numbers the following rules, established in the FIRST COURSE, apply:

(1) *To add two numbers with like signs, find the sum of their absolute values, and prefix the sign common to both.*

(2) *To add two numbers with unlike signs, find the difference of their absolute values, and prefix the sign of the one with the greater absolute value.*

Three or more positive or negative numbers may be added by adding two at a time, using rules (1) and (2). Or they may be added by the following rule, which follows from the Law of Order and the Law of Grouping:

(3) *To add three or more positive or negative numbers, find the sum of the positive numbers and the sum of the negative numbers separately, then find the sum of the sums.*

EXERCISES

Add:

$$\begin{array}{lrrrrrr} 1. & -2 & +7 & +3 & -4 & -12 & +20 \\ & \underline{-6} & \underline{+8} & \underline{-9} & \underline{+6} & \underline{+10} & \underline{-38} \end{array}$$

$$\begin{array}{lrlrlrlrlr} 2. & -5 & 3. & +3 & 4. & -8 & 5. & +1 & 6. & -16 \\ & +7 & & -9 & & -9 & & -6 & & +20 \\ & +2 & & -2 & & +6 & & -9 & & -4 \\ & -8 & & +6 & & +8 & & +7 & & -8 \\ & \underline{-4} & & \underline{7} & & \underline{-2} & & \underline{+5} & & \underline{+14} \end{array}$$

7. $-5, +4, +6, -8, -7, +2, -1, +12$.

8. Let the student make up other drills similar to these, until he is rapid and perfectly accurate in the use of the above rules.

5. Addition of Similar Terms. — Terms which have like literal parts, as $5n$ and $-7n$, or $4V^2$ and $6V^2$, are called **like** or **similar terms**. They differ only in the numerical coefficients. Terms such as *am*, *bm*, and *cm* are sometimes spoken of as "similar terms in *m*."

Similar terms are added by the following rule:

To add two or more similar terms, add their numerical coefficients, and attach to the result the common literal part.

For convenience, terms to be added may be written in a column.

Terms that are not similar (called **dissimilar** *terms) cannot be added, but their sum may be indicated by writing them in a line, connected by their given signs.*

EXERCISES

Add:

$$\begin{array}{lrlrlrlrlr} 1. & +7a & 2. & -9n & 3. & -10W & 4. & +8t^2 & 5. & -3a^2b^3 \\ & \underline{-4a} & & \underline{+2n} & & \underline{-5W} & & \underline{+4t^2} & & \underline{+5a^2b^3} \end{array}$$

$$\begin{array}{lrlrlrlrlr}
\textbf{6.} & -6N & \textbf{7.} & +8x^3 & \textbf{8.} & -2ab & \textbf{9.} & Vt & \textbf{10.} & -9y^4 \\
 & +4N & & +4x^3 & & +\ ab & & -\ Vt & & y^4 \\
 & -7N & & -9x^3 & & +4ab & & -5Vt & & 6y^4 \\
 & \underline{+3N} & & \underline{+6x^3} & & \underline{-7ab} & & \underline{9Vt} & & \underline{-5y^4}
\end{array}$$

11. $-3k, -9k, +7k, -k, -12k, 16k, -2k.$

12. Let the student make up other problems similar to these, until he is proficient in the work.

Simplify by uniting similar terms:

13. $3a^2 - 2ab - b^2 + 4ab - b^2.$

14. $7x - 8 + 3x + 5 - 4x.$

15. $A^3 - B^3 + 3A^2B - 4A^3 - 4B^3.$

16. $y^3 - 6y - 5y^2 - 8 + 2y - 4y^3 - 2.$

Indicate the addition of:

17. $7a$, $-5b$, and $-c$.

18. $3w^2$, $2wv$, and $-4v^2$.

By using the process of adding similar terms by adding their coefficients, give at sight the results of:

19. $3 \times 16 + 7 \times 16$. (Same as 10×16.)

20. $8 \times 52 + 2 \times 52$.

21. $14 \times 36 + 26 \times 36$.

22. $92 \times 48 + 8 \times 48$.

23. $36 \times 72 - 16 \times 72$.

24. $124 \times 65 - 24 \times 65$.

25. $14 \times 62 - 8 \times 62 + 24 \times 62$.

26. $81 \times 3.1416 + 19 \times 3.1416$.

27. $225 \times 3.1416 - 205 \times 3.1416$.

6. Addition of Polynomials. — Polynomials are added by the following rule:

To add polynomials, arrange the similar terms in columns, and add each column. The sums of the columns, connected by their signs, form the sum of the polynomials.

It follows from the *Law of Order* that polynomials may be rearranged, if necessary, so that the similar terms are in columns. It follows from the *Law of Grouping* that the sums of the columns, connected by their signs, form the sum of the polynomials.

EXERCISES

Add:

1. $$\begin{array}{r} 9a-5b+c \\ -2a+3b-4c \\ \underline{-5a-6b-c} \end{array}$$

2. $$\begin{array}{l} a^2-2ab+b^2 \\ a^2+2ab+b^2 \\ \underline{a^2\qquad\;\;-b^2} \end{array}$$

3. $$\begin{array}{l} x^2-2x+1 \\ x^2+5x-6 \\ \underline{x^2-8x+9} \end{array}$$

4. $$\begin{array}{lllll} a^3 & & & & -b^3 \\ & +3a^2b & -3ab^2 & & \\ \hline a^3 & + a^2b & + ab^2 & & +b^3 \end{array}$$

5. a^2-3a, $5a-6$, $2a^2+9$, and $8a-6a^2$.

6. $12-t$, $7t+3t^2$, $-4t$, and $-2t^2+16$.

7. $-2x^2+y^2$, $3xy-4y^2$, and $5x^2-xy+6y^2$.

8. a^3-1, a^2-ab+b^2, and $3-4ab+2a^2$.

9. $20-v^2$, $3v+5$, $3v^2$, and $9v-4$.

10. $\frac{1}{2}a-\frac{1}{3}b$, $\frac{1}{2}a+b$, and $a-\frac{2}{3}b$.

11. $1.5x^2-0.6x+1$, $2.5x^2+3.4x-1$, and $5-0.8x$.

12. Simplify $a^2-3a+(5a-4a^2)$.

13. If $A=x^2-y^2$, $B=x^2+2xy+y^2$, and $C=5y^2-4xy$, find $A+B+C$.

7. Subtraction of Positive and Negative Numbers. — For the subtraction of positive and negative numbers use the following rule, established in the FIRST COURSE:

To subtract, first change mentally the sign of the subtrahend, then proceed as in addition.

EXERCISES

Subtract:

1. $$\begin{array}{rrrrrr} +6 & -3 & +2 & -4 & -5 & +1 \\ \underline{+8} & \underline{-7} & \underline{-5} & \underline{+2} & \underline{-9} & \underline{-6} \end{array}$$

2. $$\begin{array}{rrrrrr} -12 & +2 & +3 & -7 & -4 & +5 \\ \underline{+15} & \underline{-10} & \underline{+9} & \underline{+8} & \underline{-1} & \underline{+9} \end{array}$$

3. $\begin{array}{rrrrrr} -24 & +20 & -12 & +\ 4 & -\ 8 & +34 \\ \underline{-30} & \underline{-16} & \underline{-18} & \underline{-36} & \underline{-22} & \underline{+27} \end{array}$

4. Subtract -8 from -6; $+3$ from -6; $+15$ from $+9$.

5. From $+12$ take -6, $+5$, $+8$, -7, -2, -12.

6. The temperature is $+4°$ at noon, and falls $10°$ by midnight. What is the temperature at midnight?

7. How many years is it from the year -250 to the year -140?

8. Cicero was born in the year -106 and assassinated in the year -43. At what age did he die?

9. Julius Cæsar was born in the year -100 and assassinated in the year -44. At what age did he die?

10. What is the meaning of $-6-(-5)$? The value?

11. Find the value of $8-(-4)$; $-2-(-8)$; $-6-(+4)$.

8. Subtraction of Similar Terms. — Since similar terms are subtracted by subtracting coefficients,

To subtract a term from a similar term, change mentally the sign of the subtrahend, then proceed as in addition of similar terms.

9. Subtraction of Polynomials. — One polynomial is subtracted from another when all of its terms are subtracted. Hence,

To subtract one polynomial from another, arrange similar terms in columns, and add to the minuend the subtrahend with the sign before each of its terms changed.

EXERCISES

Subtract:

1. $\begin{array}{rrrrr} +8a & -2V & -16x & +5t^2 & -18P^3 \\ \underline{-4a} & \underline{+7V} & \underline{-12x} & \underline{+9t^2} & \underline{+11P^3} \end{array}$

2. $\begin{array}{rrrrr} -42ab & y^2 & -4R & 3\ Ww & -\pi R^2 \\ \underline{-30ab} & \underline{-y^2} & \underline{R} & \underline{-\quad Ww} & \underline{4\pi R^2} \end{array}$

3. From $-8n^2$ take $-9n^2$; $+5n^2$; n^2; $-n^2$; $-8n^2$.

4. $7x-(-3x)=?$ $-10v-(+4v)=?$ $-24S-(-16S)=?$

Subtract:

5. $\begin{array}{r} x^2-4x+2 \\ \underline{3x^2+7x-2} \end{array}$ 6. $\begin{array}{r} 5a \quad\quad -c \\ \underline{7a-4b \quad\quad} \end{array}$ 7. $\begin{array}{r} 4a^2 \quad\quad -9 \\ \underline{-4a-10} \end{array}$

8. $2m-5n$ from $-4m-n$.

9. $-6x^3+x-5$ from $2x^3-4x^2+9x$.

10. r^4-r^2+1 from r^4+2r^2+1.

11. A^3-B^3 from $A^3-3A^2B+3AB^2-B^3$.

12. $6-4x$ from $-3x^2+5-2x^3$.

13. If $A=a^2-1$, $B=a^2-4a+4$, $C=5-3a^2$, find $A-B+C$. Find $A+B-C$.

14. $3y-z-(y-6z)$ means what? Find the value.

15. $T-4-(6-5T-2T^2)=?$

16. $a^2-b^2-(a^2-2ab+b^2)=?$

10. Removal and Insertion of Signs of Grouping. — When a sign of grouping is preceded by the sign +, it indicates addition, and when preceded by the sign −, it indicates subtraction. Hence the rules:

(1) *A sign of grouping preceded by the sign + may be removed from an expression, without changing the signs of the terms inclosed.*

(2) *A sign of grouping preceded by the sign − may be removed from an expression, if the sign before each term inclosed is changed.*

Note. — When signs of grouping are inclosed within other signs of grouping, it is best to *remove the innermost sign first, then the innermost remaining sign, etc.*

By reversing the above rules, it follows that

Terms of a polynomial may be inclosed within a sign of grouping, when this sign of grouping is preceded by the sign +, without changing the signs of the terms; and when preceded by the sign −, by changing the signs of the terms inclosed.

EXERCISES

Remove signs of grouping and combine similar terms:

1. $a - b + (2a + b)$.
2. $x + 2y - (3y - x)$.
3. $(m - n) - (m + n)$.
4. $t - [2a - (2t - 3a)]$.
5. $[5 - (x^2 - 3x) - 4x]$.
6. $(M + 6N) - \{(7M + 3N) - (7M - 3N)\}$.
7. $\{8 - (v - 3)\} - \{8 + (v - 3)\}$.
8. $a^2 - [2ab - \{b^2 - a^2\} - 2b^2] - b^2$.
9. $-W + (3g - W) - \{2W + [g - (W - g)]\} - g$.

Leave brackets, but remove parentheses and simplify within brackets:

10. $[(a - 2b) + (3a - 4b)][(a - 2b) - (3a - 4b)]$.
11. $[P + (Q - R)][P - (Q - R)]$.
12. $[(x - a) + (5y + 2a)][(x - a) - (5y + 2a)]$.

Inclose those terms containing a and b within parentheses preceded by +, and all other terms within parentheses preceded by −:

13. $a^2 - b^2 - x^2 - 2xy - y^2$.
14. $2ab - a^2 - m^2 + n^2$.
15. $a^2 - ab + b^2 - 2xy - x^2 + y^2$.
16. $a^3 - b^3 + 3x^2y + y^3 - x^3 - 3xy^2$.
17. $5b - 2a - 3r - 4s + t$.
18. $w^2v^2 - w^4 - v^4 + a^4 - b^4$.

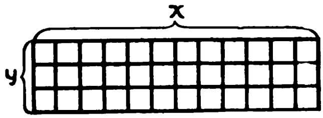
11. Fundamental Laws of Multiplication. — The product of two or more factors is the same, whatever the order in which the factors are written or taken. This truth is called the **Law of Order in Multiplication.**

The law, as applied to two factors, may be expressed by

$$xy = yx.$$

It may be illustrated graphically as follows:

The area of the rectangle is the same whether we multiply x, the number of unit squares in each row, by y, the number of rows, or multiply y, the number of unit squares in each column, by x, the number of columns.



It is evident, also, that the product of three or more factors is the same, whatever way the factors are grouped. This truth is called the **Law of Grouping in Multiplication.**

The law, as applied to three factors, may be expressed by

$$(xy)z = x(yz).$$

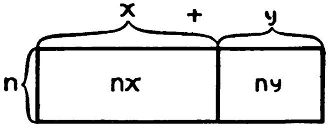
The Laws of Order and Grouping may be used to shorten arithmetical multiplication.

Thus, 2 × 7 × 8 × 5 may be written 2 × 5 × 7 × 8, by the Law of Order, and hence (2 × 5) × (7 × 8), by the Law of Grouping. Multiplying 2 by 5 gives 10, multiplying 7 by 8 gives 56, and multiplying these products gives 560. In practice, the change of order is made mentally, and factors whose product is easily found are grouped.

There is a third fundamental law of multiplication. It is evident that if the sum of two or more numbers is multiplied by a given number, the result obtained is the same as when each of those numbers is first multiplied by the given number and then the products added. This truth is called the **Law of Distribution.** It is expressed in symbols by

$$n\ (x + y) = nx + ny.$$

It may be represented graphically as follows:



The area of the entire rectangle is $n\ (x + y)$. But it is composed of two rectangles whose areas are nx and ny, respectively.

EXERCISES

By grouping factors to advantage, give at sight the products:

1. $5 \times 9 \times 2 \times 8$.
2. $2 \times 6 \times 8 \times 5$.
3. $4 \times 7 \times 5 \times 6$.
4. $8 \times 7 \times 5 \times 3$.
5. $50 \times 34 \times 2$.
6. $96 \times 25 \times 4$.
7. $8 \times 7 \times 12\frac{1}{2}$.
8. $2\frac{1}{2} \times 9 \times 4$.
9. $33\frac{1}{3} \times 8 \times 3 \times 2$.
10. $\frac{3}{8} \times 42 \times 16$.
11. $\frac{5}{9} \times \frac{2}{7} \times 21$.
12. $12\frac{1}{2} \times 70 \times 4 \times 5$
13. Show graphically that $n(a + b + c) = na + nb + nc$.
14. Show graphically that $ab = ba$.

12. Multiplication of Positive and Negative Numbers. — The following principles apply in the multiplication of positive and negative numbers:

(1) *The product of two numbers with like signs is positive.*

(2) *The product of two numbers with unlike signs is negative.*

(3) *The absolute value of the product of two numbers is the product of the absolute values of the numbers.*

From these principles the following rule is derived for finding the product of more than two factors:

Find the product of the factors regardless of signs, and prefix + or − according as the number of negative factors is even or odd.

EXERCISES

Multiply:

1. $\begin{array}{r} +6 \\ \underline{+2} \end{array}$ $\quad \begin{array}{r} -3 \\ \underline{-5} \end{array}$ $\quad \begin{array}{r} +4 \\ \underline{-8} \end{array}$ $\quad \begin{array}{r} -7 \\ \underline{+2} \end{array}$ $\quad \begin{array}{r} -3 \\ \underline{-9} \end{array}$ $\quad \begin{array}{r} +5 \\ \underline{-7} \end{array}$

2. $(-2)(-4)$, $(-7)(+8)$, $(+5)(-6)$, $(+4)(+9)$.
3. $(-7)(+6)$, $(-10)(-4)$, $(+12)(+3)$, $(-9)(+15)$.
4. $(-2)(-3)(-1)$, $(-4)(+2)(+2)$, $(+6)(-2)(-5)$.
5. $(-1)(+7)(-9)$, $(+4)(-4)(-4)$, $(-5)(-5)(-5)$.
6. $(-1)(-1)(-1)(-1)$, $(+2)(-2)(-3)(-5)(+6)(-1)$.
7. $(-2)^2$, $(-4)^2$, $(+3)^2$, $(-1)^3$, $(+2)^3$.

8. $(-1)^5$, $(-2)^2(-3)^2$, $(-5)^2(-2)^2$, $(-4)(-3)^2(-1)$.

9. Find the value of a^3 when $a = -2$; of abc when $a = -5$, $b = -1$, $c = +4$; of $a^2 + b^2$ when $a = -6$, and $b = +4$.

10. Find the value of $(x + y)(x - y)$ when $x = -4$ and $y = -9$.

3. Law of Exponents in Multiplication. — *In multiplying two powers of the same base, the exponent of the product is obtained by adding the exponents of the factors.*

In general symbols,

$$a^m \times a^n = a^{m+n}.$$

This law follows immediately from the meaning of an exponent. Thus,

$$a^m \text{ means } aaa \cdots \text{ to } m \text{ factors,}$$

and

$$a^n \text{ means } aaa \cdots \text{ to } n \text{ factors.}$$

Hence, $a^m \times a^n = aaaa \cdots \text{ to } m + n \text{ factors} = a^{m+n}$.

14. Multiplication of Monomials and Polynomials. — From the laws in the preceding sections we have the following rule:

To multiply monomials, find the product of the numerical coefficients using the laws of signs, then annex to the result the products of the like literal factors, using the law of exponents.

It follows from the Law of Distribution that

To multiply a polynomial by a monomial, multiply each term of the polynomial by the monomial, and add the partial products obtained.

It follows also from the Law of Distribution that

To multiply one polynomial by another, multiply the multiplicand by each term of the multiplier, and add all of the partial products obtained.

In multiplying one polynomial by another, the similar terms among the partial products obtained should be arranged in columns to be added.

For convenience, the terms of each of two polynomials to be multiplied should be arranged according to either the ascending or the descending powers of some letter, before multiplying.

EXERCISES

Find the products of:

1. $a^2 \times a^5$; $a^6 \times a^4$; $x^7 \times x^3$; $n^{12} \times n^4$; $P^3 \times P^8$.
2. $y \times y^2 \times y^3$; $t^4 \times t \times t^7$; $D^6 \times D^5 \times D$; $m^2 \times m^{10} \times m^6$.
3. $5a^3 \times 3a^2$; $6V^4 \times 5V^2$; $9z \times 2z^5$; $b^2 \times 7b^2 \times 4b^6$.
4. $(-3a^5)(-4a^3)$; $(+8x)(-5x^4)$; $(-2N^4)(+6N)$.
5. $(+6a^3b^2)(-4ab^4)$; $(-9m^6n)(-2mn^6)$; $(+12x^4y^4)(+4x^2y^6)$.

Find the value of:

6. $\begin{array}{l} a-b \\ \underline{a\qquad} \end{array}$ $\qquad$ $\begin{array}{l} 2m+3n \\ \underline{mn\qquad\quad} \end{array}$ $\qquad$ $\begin{array}{r} x^2-3y^2 \\ \underline{-4xy} \end{array}$

7. $12n^2(3n-5)$; $8A^4(3A^2-4A+1)$; $x^2(x^2-xy+y^2)$.
8. $5t^4(-3t^2+7t-2)$; $-3R^2(\pi R^2-2\pi RT+\pi T^2)$.
9. $(x+3)(x-7)$; $(w^2-5)(w^2-3)$; $(N-6)(N^2-4N+4)$.
10. $(a^2-1)(a^2+1)$; $(y^3+1)(y^3-1)$; $(t^4-1)(t^4-1)$.
11. $(a^2-a+1)(a^2+a+1)$.
12. $(3-s+4s^2)(2s^2+7s-5)$.
13. $(3x^2-8)(5-4x^3+2x-x^2)$.
14. $(D^2-4+4D)(3-4D^2)$.
15. $(n-1)(n^3+n^2+n+1+n^4)$.
16. $(p+1)(p+2)(p+3)$.
17. $(A+2)(A-2)(A^2+4)$.
18. $(3y-5)(9y^2+25)(5+3y)$.
19. $(x^3+v^3)(v^6+x^6)(v^3-x^3)$.

Remove signs of grouping, and combine similar terms:

20. $a(2a-1)+2a(a+1)$.
21. $5(6x-3)-8(2x+4)$.
22. $9x(y-2x)-5y(2x-7y)$.
23. $(t+5)(t-1)+(t+1)(t-5)$.
24. $(n^2-4)(n^2+3n)+(n-5)(n^3-1)$.

25. $(4\,Q-1)(Q-4)-(2\,Q+3)(3\,Q-2)$.

26. $(x^2-y^2)(x^2+y^2)-(x^2+y^2)(x^2+y^2)$.

27. $(t^3)^2=?\quad (v^4)^5=?\quad (-x^2)^4=?$

28. $(-5\,x^2y^3)^3=?\quad (6\,w^4v^7)^2=?\quad (-2\,H^6t^5)^3=?$

29. $(x+3)^2=?\quad (a-b)^2=?\quad (2\,a-3\,b)^2=?$

30. $(3\,m+5\,n)^2=?\quad (a+b+c+d)^2=?$

15. Special Products. — The products of certain **types** of expressions may be obtained by special rules found by actual multiplication.

By actual multiplication, $(2\,x^4y^3)^4=16\,x^{16}y^{12}$; $(2\,m^2n^5)^5=32\,m^{10}n^{25}$; and in general,

$$(ax^my^n)^p=a^px^{mp}y^{np}.$$

Hence, to find any power of a monomial, raise the numerical coefficient to that power and multiply each exponent in the monomial by the exponent of the power.

The above law also implies the form $((a^m)^n)^p=a^{mnp}$.

Also, by actual multiplication,

$$(a+b)(a-b)=a^2-b^2.$$

Hence, to find the product of the sum and the difference of the same two terms, take the difference between their squares.

$$(x+a)(x+b)=x^2+(a+b)\,x+ab.$$

Hence, to find the product of two binomials having a common term, take the square of the common term, plus the algebraic sum of the unlike terms multiplied by the common term, plus the algebraic product of the unlike terms.

$$(a+b)^2=a^2+2\,ab+b^2 \text{ and } (a-b)^2=a^2-2\,ab+b^2.$$

Hence, to square a binomial, take the square of the first term, plus (or minus) twice the product of the terms, plus the square of the second term.

$$(a+b+c)^2=a^2+b^2+c^2+2\,ab+2\,ac+2\,bc.$$

Hence, the square of any polynomial equals the sum of the squares of all of its terms, plus twice the algebraic product of each term into each of the terms following it.

EXERCISES

Give at sight the value of:

1. $(2^2)^3$.
2. $(10^2)^2$.
3. $(5^2)^2$.
4. $(30^2)^2$.
5. $(20^2)^3$.
6. $(2a^2b)^3$.
7. $(3a^4b^2)^2$.
8. $(10a^3b^5)^3$.
9. $(20xy^6)^2$.
10. $(30x^2y^5)^3$.
11. $(a^2b^3)^n$.
12. $(xy^2z^3)^m$.
13. $(2a^2b)^n$.
14. $((a^nb^m)^c)^d$.
15. $((2a^3b)^2)^3$.

Give at sight the products:

16. $(n+4)(n-4)$.
17. $(x-10)(x+10)$.
18. $(R+1)(R-1)$.
19. $(w-6)(w+6)$.
20. $(3-t)(3+t)$.
21. $(2a-3b)(2a+3b)$.
22. $(6r+s)(6r-s)$.
23. $(7m-2n)(2n+7m)$.
24. $(x+5)(x+2)$.
25. $T-4)(T-7)$.
26. $(V+9)(V-4)$.
27. $(y-7)(y-6)$.
28. $(2a+3)(2a+5)$.
29. $(5x-7)(5x+3)$.
30. $(12P+9)(12P-5)$.
31. $(2+8a)(7+8a)$.
32. $(x+y)^2$.
33. $(2n+5)^2$.
34. $(R-7)^2$.
35. $(5d-12)^2$.
36. $(3x+4y)^2$.
37. $(15m-2n)^2$.
38. $(1-4t^2)^2$.
39. $(9+2V)^2$.
40. $(x+y-z)^2$.
41. $(4m-3n+p)^2$.
42. $(1-2x+x^2)^2$.
43. $(4a+2b-5c)^2$.
44. $(a^2-b^2+ab)^2$.
45. $(x+y-z-w)^2$.
46. $(16t+9)(9-4t)$.
47. $(3K-7L)(7L+3K)$.
48. $(xy+13)(xy-11)$.
49. $(8M^2-1)(3+8M^2)$.
50. $(16vt-s^2)(16vt+s^2)$.
51. $(14A^3-1)(1+14A^3)$.

Give at sight the products of the following, using the principles of this section:

52. 41×41; 72×68; 97×103; 65×65; 81×84.

53. 109×101; 92×92; 64×56; 35×45; 79×79.

16. Division. — Division is the inverse of multiplication. To divide a number M (dividend) by a number N (divisor) is to find a number (quotient) which multiplied by N will give M. In general,

$$\textit{Divisor} \times \textit{Quotient} = \textit{Dividend}.$$

Hence, the principles and rules of division, deduced from the corresponding principles and rules of multiplication, are as follows:

(1) *If the dividend and divisor have like signs, the quotient is positive.*

(2) *If the dividend and divisor have unlike signs, the quotient is negative.*

(3) *In dividing powers of the same base, the exponent of the quotient is obtained by subtracting the exponent of the divisor from that of the dividend. That is,*

$$n^a \div n^b = n^{a-b}.$$

(4) *In dividing monomials, divide the coefficient of the dividend by that of the divisor, using the above laws of signs, and annex to the result the quotients of the literal factors obtained by the above law of exponents.*

(5) *To divide a polynomial by a monomial, divide each term of the dividend by the divisor, and add the partial quotients obtained.*

EXERCISES

Divide as indicated:

1. $\frac{-16}{-8}$; $\frac{+24}{+4}$; $\frac{-12}{+6}$; $\frac{+32}{-4}$; $\frac{-21}{-7}$; $\frac{+63}{-9}$.

2. $(-24) \div (-4)$; $(-42) \div (+6)$; $(+72) \div (-9)$.

3. $a^7 \div a^2$; $t^{12} \div t^7$; $V^{22} \div V^{16}$; $R^{18} \div R^6$; $n^{42} \div n^{34}$.

4. $16\,x^6 \div 2\,x^4$; $28\,P^9 \div 7\,P^5$; $96\,y^{13} \div 6\,y$; $84\,T^6 \div 12\,T^3$.

5. $(-26\,a^{14}) \div (-2\,a^9)$; $(+32\,V^{16}) \div (-8\,V^{12})$.

6. $(+100\,x^3y^7) \div (-20\,xy^4)$; $(-a^{10}b^3) \div (-ab^2)$.

7. $4\,x^2\overline{)16\,x^4 - 8\,x^3}$; $-5\,m^2n\overline{)15\,m^4n - 10\,m^3n^2 + 20\,m^2n^3}$.

8. $(4\,a^2b^4 - 12\,ab^5 - 6\,a^3b^3) \div (-2\,ab^3)$.

9. $(21\,r^4s^2 + 28\,rs^5 - 35\,r^3s^3) \div (-7\,rs^2)$.

10. $(-56\,W^5 - 49\,W^4V + 63\,W^3V^2 - 21\,W^2V^3) \div (-7\,W^2)$.

11. $(42\,x^4y^4 - 36\,x^2y^6 - 48\,x^6y^2) \div (+6\,x^2y^2)$.

12. $(\frac{1}{2}\,M^3N - \frac{1}{4}\,M^2N^2 + \frac{1}{2}\,MN^3) \div (\frac{1}{2}\,MN)$.

17. Division of One Polynomial by Another. — The method of dividing one polynomial by another, as shown in the FIRST COURSE, is as follows:

(1) *Arrange both the dividend and divisor according to the descending or ascending powers of some letter.*

(2) *Divide the first term of the dividend by the first term of the divisor to obtain the first term of the quotient.*

(3) *Multiply the whole divisor by this term of the quotient, and subtract the result from the dividend.*

(4) *Treat the remainder as a new dividend (having the terms arranged as before), and repeat the process, continuing until either the remainder zero, or a true remainder, is found.*

In arranging the terms for division, if terms containing powers of the letter of arrangement between the highest and lowest powers of that letter in the dividend are missing, spaces should be left for them in the written work, as shown in the illustration below.

$$
\begin{array}{r|llll}
 & a^2 & - ab & + b^2 & \\
\cline{2-5}
a + b & a^3 & & & + b^3 \\
 & \underline{a^3} & \underline{+ a^2b} & & \\
 & & - a^2b & & + b^3 \\
 & & \underline{- a^2b} & \underline{- ab^2} & \\
 & & & ab^2 & + b^3 \\
 & & & \underline{ab^2} & \underline{+ b^3}
\end{array}
$$

EXERCISES

Divide:

1. $n^2 + 7n + 10$ by $n + 2$.
2. $x^2 - 12x + 32$ by $x - 8$.
3. $54 + 3a - a^2$ by $9 - a$.
4. $4t - 60 + t^2$ by $t + 10$.
5. $y^2 - 16$ by $y - 4$.
6. $9 - v^2$ by $3 + v$.
7. $1 - 25N^2$ by $5N + 1$.
8. $R - 72 + R^2$ by $R - 8$.
9. $a^3 - 1$ by $a - 1$.
10. $27 + m^3$ by $m + 3$.
11. $1 - x^4$ by $1 - x$.
12. $s^3 + t^3$ by $s^2 - st + t^2$.
13. $A^4 + A^2B^2 + B^4$ by $A^2 + AB + B^2$.
14. $8b^3 + 125 - 30ab + a^3$ by $a + 5 + 2b$.

18. Synthetic Division. — An abbreviated method of division, called **synthetic division**, may be used to advantage when the dividend in a polynomial is some letter, say x, and the divisor of the form $x - a$.

Thus, the usual process of division of $x^3 - 9x^2 + 27x - 40$ by $x - 5$, bringing down only such terms in each remainder as are immediately needed, is as follows:

$$\begin{array}{rl}
 & \underline{x^2 - 4x + 7\qquad\qquad\quad} \\
x - 5) & x^3 - 9x^2 + 27x - 40 \\
 & \underline{x^3 - 5x^2} \\
 & -4x^2 + 27x \\
 & \underline{-4x^2 + 20x} \\
 & \qquad\qquad 7x - 40 \\
 & \qquad\qquad \underline{7x - 35} \\
 & \qquad\qquad\quad -5, \text{ rem.}
\end{array}$$

Observe that the coefficients of the terms in the quotient are the coefficients of the first term of the dividend and of each of the remainders, taken in order.

Since the first term of each partial product cancels the term just above it in the subtraction, the former need not have been written.

Hence, if the dividend is written in descending powers of x, the first term of the divisor need not be written at all.

If the sign of each term of the partial products had been changed, this term might have been added to the term above it, instead of subtracted from it. The same result may be obtained by changing the sign of each term of the divisor, multiplying, and adding the partial products to the terms above them.

Hence, omitting all work that is unnecessary, writing the coefficients only, and raising all of the remainders into one line, the division may be written as follows:

$$\begin{array}{r|rrrr} 5) & 1 & -\ 9 & +\ 27 & -\ 40 \\ & & +\ 5 & -\ 20 & +\ 35 \\ \hline & 1 & -\ 4 & +\ 7 & -\ 5 \end{array}$$

where 1, − 4, and + 7 are the coefficients of the quotient, and − 5 the remainder.

Thus we see that in **synthetic division**:

The first term of the divisor is omitted, the sign of the second term of the divisor changed, and only coefficients of the dividend, partial products, remainders, and quotient, written.

The first coefficient of the dividend is brought down for the first coefficient of the quotient.

This number is multiplied by the number in the divisor and added to the next term of the dividend for the second coefficient of the quotient, and so on to the last term. The last sum is the true remainder.

Example. — Divide $3x^3 - 7x^2 + 13$ by $x + 2$.

The 2 of the divisor is placed at the left, with its sign changed. Since the first power of x is missing from the dividend, the coefficient 0 must be supplied. The first coefficient 3 is brought down for the first term of the quotient. Then $(-2)(3) = -6$; $-7 - 6 = -13$; $(-2)(-13) = +26$; $0 + 26 = +26$; $(-2)(+26) = -52$; $+13 - 52 = -39$. Hence the quotient is $3x^2 - 13x + 26$, and remainder -39.

$$\begin{array}{r|rrrr} -2) & 3 & -\ 7 & +\ 0 & +\ 13 \\ & & -\ 6 & +\ 26 & -\ 52 \\ \hline & 3 & -\ 13 & +\ 26 & -\ 39 \end{array}$$

If any power of x in the dividend is missing, its place must be supplied by a zero.

EXERCISES

Divide by the synthetic method:

1. $x^3 + 3x^2 - 7x - 3$ by $x - 2$.
2. $2x^4 - 9x^2 + 3x - 5$ by $x - 4$.
3. $a^4 - 3a^2 + 7a - 1$ by $a - 2$.
4. $3n^3 + 7n^2 - 3n + 15$ by $n + 3$.
5. $y^4 - 7y^3 + 2y - 12$ by $y + 2$.
6. $t^3 + 2t^2 + 96$ by $t + 4$.
7. $V^3 + 3V^2 + V - 7$ by $V - 3$.
8. $2r^3 + r^2 - 7$ by $r + 3$.
9. $A^4 + 5A^3 + A - 8$ by $A + 4$.
10. $n^4 + 3n^2 - 9$ by $n + 6$.

SUPPLEMENTARY EXERCISES

Note. — These exercises and the supplementary exercises at the ends of other chapters may be used by all classes for reviews and for more difficult exercises than those contained in the body of the text. They may be omitted, however, without interfering with the continuity of the work.

1. Express $BBB \cdots$ to n factors by use of an exponent.

2. Express t^a without an exponent.

By adding the literal coefficients of the like literal parts, find the sums of:

3. an, bn, and cn.

Solution. — Indicating the addition of a, b, and c, the sum is $(a+b+c)n$.

4. ax, $-bx$, and cx.

5. mr^2, nr^2, and $-pr^2$.

6. $2ay$, $-3by$, and cy.

7. $4ax^2y$, $9nx^2y$, and $-5bx^2y$.

8. $ax + by + cz$, $bx + cy + az$, and $cx + ay + bz$.

9. From $5m - 2n$ subtract $n - 4p$.

10. From $Ax - By$ subtract $Cx + Dy$.

11. From $2ax + 4cz$ subtract $5by - az$.

12. Simplify $-2 - [a + \{5 - (3a - 1) + a\} - 1] - a$.

13. Remove parentheses, leaving brackets, and simplify
$$[(5x - y) + (2y - 4z)][(5x - y) - (2y - 4z)].$$

Multiply:

14. x^n, x^{4n}, and x^{3n}.

15. $a^{n+1}b$, ab^{n-1}, and a^2b^2.

16. R^{a-4}, $2R^{2a+1}$, and $-3R^{5-a}$.

17. $A^x + B^x$ and $A^x - B^x$.

18. $a^n - b^n$, $a^n + b^n$, $a^{2n} + b^{2n}$.

19. $T^{2a} + t^{2b}$, $T^{2a} - t^{2b}$, $T^{4a} + t^{4b}$.

20. $x^n \cdot x^n \cdot x^n \cdots$ to 8 factors.

21. $x^n \cdot x^n \cdot x^n \cdots$ to m factors.

22. $(x^ay^b)(x^ay^b) \cdots$ to n factors.

23. $(2a^nb)(2a^nb) \cdots$ to c factors.

Divide:

24. $M^{'a}$ by M^a; B^{3x-1} by B^{x-3}.

25. $-48x^{3m}y^n$ by $-4x^ny^{3m}$.

26. $32W^{a+2}V^{a-2}$ by $4W^6V^{a-4}$.

27. $N^b - 3N^{b+1} + 2N^{b+2}$ by N^b.

28. $x^{2n} - y^{2n}$ by $x^n - y^n$.

29. $A^{4n} - B^{4n}$ by $A^n - B^n$.

30. Simplify $-n[a - n(3a - n) - n^2] + an$.

CHAPTER II

FACTORS AND MULTIPLES

19. Factoring. — Those expressions whose product is a given expression are called the **factors** of the given expression. The process of finding the factors of a given expression is called **factoring**. If a factor of an expression cannot itself be factored, it is called a **prime factor**. In general, to factor an expression means to find its prime factors.

Ability to factor an expression depends upon recognition of the **special type form** of the expression. The methods of factoring expressions of certain *special type forms* that were discussed in the FIRST COURSE are restated in §§ 20–23.

20. Monomial Factors. — In factoring any expression, if it has a monomial factor, this factor should be found first, and the expression written as the product of the monomial and a polynomial factor. The polynomial factor should then be factored if possible. To find the monomial and polynomial factors of an expression,

Find, by inspection, the monomial of highest power that will divide each term of the expression. Divide the expression by this monomial. The divisor and quotient are the monomial and polynomial factors, respectively, of the given expression.

EXAMPLE. — Factor $3\,m^4n - 9\,m^3n^2 + 3\,m^2n^3$.

Dividing each term by $3\,m^2n$,

$$3\,m^4n - 9\,m^3n^2 + 3\,m^2n^3 = 3\,m^2n(m^2 - 3\,mn + n^2).$$

EXERCISES

Factor:

1. $6\,a^4 - 18\,a^3$.
2. $15\,m^3n + 12\,m^2n^2 + 9\,mn^3$.
3. $4\,x^6 - 14\,x^4y^2 + 10\,x^3y^3$.
4. $2\,\pi R^2 + 4\,\pi RH$.

5. $7t^4 - 28t^3v - 14t^2v^2$.
6. $-A^3 - A^2 - A$.
7. $-a^2 - 2ab - b^2$.
8. $x^4y^3z^6 + x^5y^4z^4 + x^3y^5z^5$.
9. $4\pi R^2 + 4\pi r^2 + 4\pi Rr$.
10. $8V_1^2 - 16V_1V_2 + 8V_1V_3$.
11. $3m^4n - 6m^3n^2 + 3m^2n^3$.
12. $x^n + ax^n$.
13. $a^nb^n - a^n$.
14. $t^2v^{n+2} + t^{n+2}v^2$.
15. $P^{n+1} + P^nQ$.
16. $5a^ny^{n-1} + 10a^{n-1}y^n$.

21. Factoring Binomials. — To factor a binomial, see if it is of either of the following special forms, and determine the factors accordingly.

I. A Binomial the Difference of Two Squares

By multiplication,

$$a^2 - b^2 = (a + b)(a - b).$$

Hence, the factors of the difference of the squares of two numbers are the sum and the difference of the numbers.

Example. — $64V^2 - 81T^2 = (8V)^2 - (9T)^2$
$= (8V + 9T)(8V - 9T)$.

II. A Binomial the Difference of Two Cubes

By multiplication,

$$a^3 - b^3 = (a - b)(a^2 + ab + b^2).$$

Hence, one factor of the difference of the cubes of two numbers is the difference of the numbers, and the other is the sum of their squares and their product.

Example. — $8R^3 - 27D^3 = (2R)^3 - (3D)^3$
$= (2R - 3D)(4R^2 + 6RD + 9D^2)$.

III. A Binomial the Sum of Two Cubes

By multiplication,

$$a^3 + b^3 = (a + b)(a^2 - ab + b^2).$$

Hence, one factor of the sum of the cubes of two numbers is the sum of the numbers, and the other is the sum of their squares minus their product.

Example. — $64\,n^{15} + 1 = (4\,n^5)^3 + 1$
$= (4\,n^5 + 1)(16\,n^{10} - 4\,n^5 + 1).$

A binomial may be one of the above forms, and then when factored, one or both of its factors may be some one of these forms. In such a case the factors should be factored.

Example. — $x^6 - 1 = (x^3)^2 - 1$
$= (x^3 + 1)(x^3 - 1)$
$= (x + 1)(x^2 - x + 1)(x - 1)(x^2 + x + 1).$

EXERCISES

Factor:

1. $9\,A^2 - 16.$
2. $4\,t^2 - 25.$
3. $1 - 49\,n^2.$
4. $25\,x^2 - 64\,y^2.$
5. $81\,V^2 - 1.$
6. $4\,P^2 - 9\,Q^2.$
7. $25\,R^2 - 16.$
8. $36\,v^2 - 49\,w^2.$
9. $a^3 - 8\,b^3.$
10. $M^3 - 27.$
11. $1 - 64\,k^3.$
12. $x^3 - 125\,y^3.$
13. $216 - T^3.$
14. $m^3 + 27\,n^3.$
15. $A^3 + 1.$
16. $8\,r^3 + 1.$
17. $1 + 64\,P^3.$
18. $125\,p^3 + 8\,q^3.$
19. $8\,W^3 - 1.$
20. $27 + s^3.$
21. $3\,a^2 - 12\,b^2.$
22. $5 - 40\,N^3.$
23. $48\,x^3 + 6\,y^3.$
24. $x^4 - 1.$
25. $y^6 - z^6.$
26. $A^8 - B^8.$
27. $1 - t^6.$
28. $m^6 + n^6.$
29. $x^{12} - y^{12}.$
30. $am^4 - an^4.$
31. $p^6q^3 + p^3q^6.$
32. $\pi R^3 - \pi RH^2.$
33. $8\,v^4t - 64\,vt^4.$
34. $(a + b)^2 - c^2.$
35. $(m - n)^2 - x^2.$
36. $p^2 - (q - r)^2.$
37. $(v + t)^2 - (w + u)^2.$
38. $(M - N)^2 - (P - Q)^2.$
39. $(2\,A + B)^2 - (C - 3\,D)^2.$

22. Factoring Trinomials. — To factor a trinomial, see if it is of either of the following special forms, and determine the factors accordingly.

I. Trinomials that are Perfect Squares

By multiplication,

$$(a + b)^2 = a^2 + 2\,ab + b^2 \text{ and } (a - b)^2 = a^2 - 2\,ab + b^2.$$

Evidently, a trinomial is the square of a binomial when two of its terms are squares and the other term twice the product of their square roots.

To factor such a trinomial express it as the square of a binomial. It follows from the above identities that:

To find the binomial of which a given trinomial is a square, take as the terms the square roots of the two square terms of the trinomial, and connect them by the sign of the other term.

EXAMPLE. — $25M^2 - 30MN + 9N^2 = (5M - 3N)^2$.

II. TRINOMIALS OF THE FORM $x^2 + ax + b$

Since, by § 15, the product of two binomials having a common term x must be of the form $x^2 + ax + b$, in which a is the algebraic sum of the unlike terms of the binomials and b their product, it follows that:

To factor a trinomial of the form $x^2 + ax + b$, *take as factors two binomials with a common term* x *and having for their other terms two numbers whose algebraic sum is* a *and algebraic product* b.

EXAMPLE. — Factor $R^2 - 4R - 12$.

For the second terms of the factors we must find two numbers whose sum is -4 and whose product is -12. These are evidently -6 and $+2$. Hence,

$$R^2 - 4R - 12 = (R-6)(R+2).$$

III. TRINOMIALS OF THE FORM $ax^2 + bx + c$

If a trinomial of the form $ax^2 + bx + c$ has binomial factors, they are of the forms $mx + p$ and $nx + q$. Of the various methods of factoring such a trinomial, there is none better than the following:

Find by trial two binomials whose product is the given binomial.

EXAMPLE. — Factor $4A^2 - 3A - 27$.

The product of the first terms of the factors must be $4A^2$. Hence, they must be either A and $4A$, or $2A$ and $2A$. The product of the last terms must be -27. Hence, they must have unlike signs, and must be either 1 and

-27, -1 and 27, 3 and -9, or -3 and 9. Hence, the possible trial sets of factors are:

$$\begin{array}{ccccc} A+1 & A-1 & A+3 & A-3 & 2A+1 \\ \underline{4A-27} & \underline{4A+27} & \underline{4A-9} & \underline{4A+9} & \underline{2A-27 \text{ etc.}} \end{array}$$

Only the fourth of these gives the right sum of cross products, $-3A$. Hence,

$$4A^2-3A-27=(A-3)(4A+9).$$

EXERCISES

Factor:

1. $x^2-10x+25$.
2. $4a^2+4a+1$.
3. $9t^2-24tv+16v^2$.
4. $1+14P+49P^2$.
5. $R^2-12Rr+36r^2$.
6. $n^2+7n+10$.
7. $x^2-11x+18$.
8. A^2-A-56.
9. $w^2+2wv-35v^2$.
10. $T^2-7TU-30U^2$.
11. $8a^2-10a+3$.
12. $2V^2+7V-30$.
13. $15M^2-2M-8$.
14. $12R^2+25Rr+12r^2$.
15. $10x^2+29xy-21y^2$.
16. $64p^2+48pq+9q^2$.
17. $3W-27+4W^2$.
18. $L^2+5Lv-14v^2$.
19. $mn^2-3mn-54m$.
20. $5nx^2+10nxy^2+5ny^4$.
21. $4aF^2+8aF-60a$.
22. $4k^2n-2kmn-20m^2n$.
23. $-B^3-2B^2-B$.
24. $6h^2m^2+17h^2m-14h^2$.
25. $5r_1^2-12r_1r_2+4r_2^2$.
26. $36-30D-6D^2$.
27. $tT^2-30tTv+225tv^2$.
28. $-4-24d^2-36d^4$.
29. $4\pi R^2+28\pi-22\pi R$.
30. $45V_1^2+12V_2^2+69V_1V_2$.
31. $(x+b)^2-5(x+b)+6$.
32. $(m-n)^2+2(m-n)(w+v)+(w+v)^2$.
33. $4(A+B)^2+3C(A+B)-27C^2$.

23. Factoring Polynomials by Grouping Terms. — By grouping terms, some polynomials may be written in one of the special forms given in §§ 20, 21, and 22, and factored accordingly.

I. POLYNOMIALS WHOSE TERMS MAY BE GROUPED TO GIVE A COMMON FACTOR

Group the terms of the polynomial to be factored so that a factor may be divided out of each group, and that the expressions left in parentheses are the same in all of the groups. Then divide by the expression in parentheses, writing the divisor as one factor and the quotient as the other.

EXAMPLE 1. — Factor $2v + v^2 - 2w - vw$.

Grouping the first and second terms, and also the third and fourth terms, and dividing out the monomial factor of each group,

$$\begin{aligned} 2v + v^2 - 2w - vw &= v(2+v) - w(2+v) \\ &= (2+v)(v-w). \end{aligned}$$

EXAMPLE 2. — Factor $a^2 - b^2 - ac + bc$.

Grouping the first two terms, and also the last two, and factoring the groups,

$$\begin{aligned} a^2 - b^2 - ac + bc &= (a+b)(a-b) - c(a-b) \\ &= (a-b)(a+b-c). \end{aligned}$$

II. POLYNOMIALS WRITTEN AS THE DIFFERENCE OF TWO SQUARES

Group those terms that form the square of a binomial, and express the polynomial in the form $a^2 - b^2$.

EXAMPLE. — Factor $4a^2 - b^2 + 6bc - 9c^2$.

Grouping the last three terms in parentheses preceded by the sign −,

$$\begin{aligned} 4a^2 - b^2 + 6bc - 9c^2 &= 4a^2 - (b^2 - 6bc + 9c^2) \\ &= 4a^2 - (b - 3c)^2 \\ &= (2a + [b - 3c])(2a - [b - 3c]) \\ &= (2a + b - 3c)(2a - b + 3c). \end{aligned}$$

EXERCISES

Factor:

1. $a^2 - 2ab + b^2 - c^2$.
2. $x^2 - y^2 - 2y - 1$.
3. $an - bn - am + bm$.
4. $y^2 + xy - yz - xz$.
5. $xP - yQ - xQ + yP$.
6. $25A^2 + 2BC - B^2 - C^2$.
7. $m^2 + n^2 - 2mn - v^2 - t^2 - 2vt$.
8. $B^2 - 4AC - A^2 - 4C^2$.

9. $sP - 5s + 5t - tP$.

10. $na - nb + a - b$.

11. $V^2 - V - Vv + v$.

12. $R^2 - r^2 - r + R$.

13. $1 + 4y + 4y^2 - z^2$.

14. $1 - 6b + 4mn + 9b^2 - m^2 - 4n^2$.

15. $H^2 - 16t^2 + Hv - 4vt$.

16. $2a - 2b + ad - bd + 2c + cd$.

17. $am^2 + am + bm^2 + bm + a + b$.

18. $x^2 - y^2 + z^2 - w^2 - 2(xz - yw)$.

24. Trinomials of the Form $a^4 + na^2b^2 + b^4$. — Some trinomials of the special form $a^4 + na^2b^2 + b^4$, which cannot be factored by the methods of §22, may be written in the form $a^2 - b^2$, by the addition and subtraction of a term and by grouping terms as in § 23.

Add such a positive term to the middle term as will make the trinomial a perfect square, and indicate the subtraction of the term added.

Example. — Factor $4x^4 - 16x^2y^2 + 9y^4$.

By adding and indicating the subtraction of $4x^2y^2$,

$$\begin{aligned} 4x^4 - 16x^2y^2 + 9y^4 &= 4x^4 - 12x^2y^2 + 9y^4 - 4x^2y^2 \\ &= (2x^2 - 3y^2)^2 - (2xy)^2 \\ &= (2x^2 - 3y^2 + 2xy)(2x^2 - 3y^2 - 2xy). \end{aligned}$$

EXERCISES

1. $n^4 + n^2 + 1$.

2. $1 + 3a^2 + 4a^4$.

3. $x^8 + x^4 + 1$.

4. $t^4 + 2t^2v^2 + 9v^4$.

5. $V^4 - 3V^2 + 9$.

6. $4a^4 + 16a^2b^2 + 25b^4$.

7. $4M^4 + 4M^2N^2 + 25N^4$.

8. $R^4 - 15R^2r^2 + 9r^4$.

9. $36A^4 - 76A^2T^2 + 25T^4$.

10. $49y^8 + 19y^4z^2 + 4z^4$.

11. $4W^8 - 16W^4w^6 + 25w^{12}$.

12. $64a^4b^4 + 12a^2b^2 + 1$.

13. $m^8 - 22m^4x^2y^4 + 9x^4y^8$.

14. $8\pi R^4 - 90\pi R^2r^2 + 50\pi r^4$.

15. $4v^{12}t - 4v^6t^3 + 64t^5$

16. $75a^5 + 48a^3b^4c^2 + 12ab^8c^4$.

25. Factoring by the Remainder Theorem. — A polynomial that cannot be factored by any of the methods in the preceding sections may sometimes be factored by use of the following principle, called the **Remainder Theorem**.

If a polynomial in x be divided by $x - a$, the remainder is the same as the dividend when x is replaced by a.

Thus, dividing $x^3 - x^2 + 3x + 2$ by $x - 2$ gives a remainder 12. Now, if x is replaced by 2, the dividend becomes $8 - 4 + 6 + 2$, or 12. Hence, the remainder is the same as the dividend when x is made 2.

$$\begin{array}{r} x^2 + x + 5 \\ x - 2)\overline{x^3 - x^2 + 3x + 2} \\ \underline{x^3 - 2x^2} \\ x^2 + 3x \\ \underline{x^2 - 2x} \\ 5x + 2 \\ \underline{5x - 10} \\ 12, \text{ rem.} \end{array}$$

To show that this principle is true in general, let any polynomial in x be represented by the general expression:

$$Ax^n + Bx^{n-1} + Cx^{n-2} + \cdots + Mx + N.$$

Let the quotient and remainder, when this is divided by $x - a$, be represented by Q and R respectively. Then since

$$\textit{Dividend} = \textit{Divisor} \times \textit{Quotient} + \textit{Remainder},$$

$$Ax^n + Bx^{n-1} + Cx^{n-2} + \cdots + Mx + N = (x - a)Q + R.$$

Since x may be any number, this equation must be true when $x = a$. But when $x = a$, it becomes

$$Aa^n + Ba^{n-1} + Ca^{n-2} + \cdots + Ma + N = (a - a)Q + R.$$

Hence, since the term $(a - a)Q$ is zero,

$$Aa^n + Ba^{n-1} + Ca^{n-2} + \cdots + Ma + N = R;$$

which establishes the general principle.

Now, since $x - a$ is a *factor* of the polynomial when the remainder is zero, the following **factor theorem** may be stated:

If a polynomial in x becomes zero when x is replaced by a number a, then $x - a$ is a factor of the polynomial.

EXAMPLE. — Factor $x^3 - 5x^2 + 10x - 12$.

If x is replaced by 3, the polynomial becomes $27 - 45 + 30 - 12$, or 0. Hence, $x - 3$ is a factor.

Dividing the polynomial by $x - 3$ gives the quotient $x^2 - 2x + 4$ for the other factor.

$$\begin{array}{r} 3)1 - 5 + 10 - 12 \\ \underline{+ 3 - 6 + 12} \\ 1 - 2 + 4 \end{array}$$

Now $x^2 - 2x + 4$ is of the form $x^2 + ax + b$, but cannot be factored.

Hence, $x^3 - 5x^2 + 10x - 12 = (x - 3)(x^2 - 2x + 4)$.

Observe that, in the above example, since the product of the last terms of the factors must be -12, the only values that should be substituted for x in the given expression are divisors of -12. In general, when an expression to be factored is written in descending powers of the letter of arrangement, only divisors of the last term should be used for the letter of arrangement in discovering the factors.

When a binomial factor has been discovered by the Remainder Theorem, the quickest way to divide it out is by *synthetic division*, as shown in the above example.

If a polynomial in x has more than one factor of the form $x - a$, these may *all* be found by trial before any division is performed, and the polynomial be divided by their product to obtain the remaining factor. In some problems the division may not be necessary at all.

EXAMPLE. — Factor $x^3 - 5x^2y - 8xy^2 + 12y^3$.

The only values that may be substituted for x are y, $-y$, $2y$, $-2y$, $3y$, $-3y$, $4y$, $-4y$, $6y$, $-6y$, $12y$, or $-12y$. By trial it is found that when x is replaced by y, $-2y$, or $6y$, the given expression becomes zero. Hence, three factors are $x - y$, $x + 2y$, and $x - 6y$. It is evident that there can be only these three factors. Hence,

$$x^3 - 5x^2y - 8xy^2 + 12y^3 = (x - y)(x + 2y)(x - 6y).$$

EXERCISES

Factor by the Remainder Theorem:

1. $2x^2 - x - 1$.
2. $3x^2 + 5x + 2$.
3. $2y^2 - 5y + 2$.
4. $4n^2 + 7n - 2$.
5. $2t^2 - 9t + 9$.
6. $3A^2 + 14A + 15$.
7. $3V^2 - 13V - 10$.
8. $x^4 - 3x^2 - 4$.
9. $a^3 - 21a - 20$.
10. $y^3 - 6y^2 + 11y - 6$.
11. $b^3 + b^2 - 3b - 2$.
12. $P^3 + 7P^2 + 7P - 15$.
13. $n^4 + n^3 - 2n - 2$.
14. $2v^3 + 3v^2 - 50v + 24$.
15. $3T^3 - 2T^2 + T - 2$.
16. $a^3 + 3a^2 + 3a + 2$.
17. $R^3 + 2R^2 - 4R - 3$.
18. $s^4 + 7s^3 + 13s^2 + 7s + 12$.
19. $z^6 - z^4 + z^2 - 1$.
20. $a^3 + 2a^2b - ab^2 - 2b^3$.
21. $2x^2 - 3xy + y^2$.

22. $3x^2 + 2xy - y^2$.

23. $m^3 + 2m^2y - my^2 - 2y^3$.

24. $s^3 - 2s^2y - 9sy^2 + 18y^3$.

25. $t^3 + 4t^2d + td^2 - 6d^3$.

26. $A^3 - 3A^2B + 3AB^2 - B^3$.

27. $N^3 - 7a^2N - 6a^3$.

28. Show that $a - b$ is a factor of $a^{20} - b^{20}$.

29. Show that $a + b$ is a factor of $a^{15} + b^{15}$.

26. Binomials of the Forms $a^n - b^n$ and $a^n + b^n$.—The factors of $a^n - b^n$ when n is 2, and of both $a^n - b^n$ and $a^n + b^n$ when n is 3, have been found in § 21. The factors of $a^n - b^n$ and $a^n + b^n$ for all values of n may be found by the Remainder Theorem.

In $a^n - b^n$, let $a = b$. Then $a^n - b^n$ becomes $b^n - b^n$ or 0.

(1) *Hence, $a - b$ is a factor of $a - b^n$ for all values of n.*

In $a^n - b^n$, let $a = -b$. Then $a^n - b^n$ becomes $(-b)^n - b^n$, which is 0 when n is even, but not 0 when n is odd.

(2) *Hence, $a + b$ is a factor of $a^n - b^n$ only when n is even.*

In $a^n + b^n$, let $a = b$. Then $a^n + b^n$ becomes $b^n + b^n$, which is not 0 for any value of n.

(3) *Hence, $a - b$ is not a factor of $a^n + b^n$ for any value of n.*

In $a^n + b^n$, let $a = -b$. Then $a^n + b^n$ becomes $(-b)^n + b^n$, which is 0 when n is odd, but not 0 when n is even.

(4) *Hence, $a + b$ is a factor of $a^n + b^n$ only when n is odd.*

The polynomial factors of $a^n - b^n$ and $a^n + b^n$ for special values of n are found by division. Thus, by division,

$a^5 - b^5 = (a - b)(a^4 + a^3b + a^2b^2 + ab^3 + b^4)$;

$a^6 - b^6 = (a - b)(a^5 + a^4b + a^3b^2 + a^2b^3 + ab^4 + b^5)$;

$a^7 + b^7 = (a + b)(a^6 - a^5b + a^4b^2 - a^3b^3 + a^2b^4 - ab^5 + b^6)$;

$a^8 - b^8 = (a + b)(a^7 - a^6b + a^5b^2 - a^4b^3 + a^3b^4 - a^2b^5 + ab^6 - b^7)$; etc.

In general,

When the binomial factor is $a - b$, the signs of all terms of the polynomial factor are $+$; and when the binomial factor is $a + b$, the signs of the successive terms of the polynomial factor are alternately $+$ and $-$.

In every polynomial factor the exponents of a *diminish by* 1 *and the exponents of* b *increase by* 1, *as we proceed term by term from left to right.*

Example. — Factor $x^{10} - 32y^5$.

Indicating the powers of the factors in the terms, and simplifying afterward,

$$\begin{aligned}x^{10} - 32y^5 &= (x^2)^5 - (2y)^5\\ &= (x^2 - 2y)[(x^2)^4 + (x^2)^3(2y) + (x^2)^2(2y)^2 + (x^2)(2y)^3 + (2y)^4]\\ &= (x^2 - 2y)(x^8 + 2x^6y + 4x^4y^2 + 8x^2y^3 + 16y^4).\end{aligned}$$

EXERCISES

Factor:

1. $a^7 - b^7$.
2. $x^7 + 1$.
3. $a^5 - 32b^5$.
4. $1 - t^5$.
5. $1 - n^7$.
6. $a^{11} - b^{11}$.
7. $a^{11} + b^{11}$.
8. $R^{13} + r^{13}$.
9. $R^{13} - r^{13}$.
10. $32x^5 + 243y^5$.
11. $2 - 256B^7$.
12. $m^{10} + n^5$.
13. $x^6 + y^6$.

Suggestion. — Write it as the sum of odd powers, thus: $(x^2)^3 + (y^2)^3$.

14. $1 + a^{10}$.
15. $x^{12} + y^{12}$.
16. $n^{14} + 1$.
17. $v^{18} + w^{18}$.
18. $5 + 5P^{10}$.
19. $4 + 4N^6$.
20. $a^{15} + b^{15}$.
21. $1 + y^{25}$.
22. $a^6 - 1$.

Suggestion. — Write it as the difference of two squares, thus, $(a^3)^2 - 1$.

23. $x^{12} - y^{12}$.
24. $m^{10} - n^{10}$.
25. $1 - t^{16}$.
26. $256 - a^6$.
27. $a^{14} - 1$.
28. $p^8 - q^8$.
29. $w^{12} - v^6$.
30. $1 - P^9$.

27. General Directions for Factoring. — In attempting to factor any given expression,

(1) *First see if there is a monomial factor, and if there is, factor the expression into its monomial factor and the corresponding polynomial factor.*

(2) *Then, to factor the polynomial factor, see if it is of either one of the special type forms discussed in* §§ 21, 22, 23, 24, *and* 26; *and if so, use the method applicable to that special form.*

(3) *If unsuccessful, try the Remainder Theorem.*

(4) *Factor each polynomial factor obtained, if possible, until the expression is resolved into its prime factors.*

MISCELLANEOUS EXERCISES

Find the prime factors of:

1. $2N^2 - N - 1$.
2. $ax^4 - 64a^5$.
3. $3abn^2 + 2abn - ab$.
4. $4P^2 + 32PQ + 39Q^2$.
5. $bt^2 - 6b^2t + 5t^3$.
6. $2V^2 - 3VW + W^2$.
7. $x^4 + 3x^2 - 4$.
8. $xy + z^2 - xz - yz$.
9. $n^3 + 3n^2 + 6n + 18$.
10. $4(a - b)^2 - (y + v)^2$.
11. $9R^2 - 27Rr + 20r^2$.
12. $B^4 + B^2C^2 + C^4$.
13. $x^4 - 18x^2y^2 + y^4$.
14. $T^6 + T^3 - 42$.
15. $h^4 + 2h^2k^2 - 35k^4$.
16. $y^2 + 3y^3 - y^4 - 3y$.
17. $1 + bt - (x^2 + bx)t^2$.
18. $(x^2 + y^2 - z^2)^2 - 4x^2y^2$.
19. $V^4 - 2Vv^3 - v^4 + 2V^3v$.
20. $ay^2 + 7ay - 30a$.
21. $ac + cd - ab - bd$.
22. $p^4 - (p - 6)^2$.
23. $72t^2 + 41t - 45$.
24. $a^2b^2 - a^2 - b^2 + 1$.
25. $2z^2 + 5zt - 12t^2$.
26. $6a^3N - a^2N^2 - 9a^4$.
27. $1 - 16m^8$.
28. $a^2 - b^2 - 3a + 3b$.
29. $bcn + acn^2 + bn^3 + an^4$.
30. $a^4 - 2(b^2 + c^2)a^2 + (b^2 + c^2)^2$.
31. $x^2 - (4a^2 + b^2)a^2x + 4a^6b^2$.
32. $n^2 + (x - y)n - 2x(x + y)$.
33. $4w^4 + 3w^2v^2 + 9v^4$.
34. $A^2B - BC^2 + A^2C - C^3$.
35. $K^3 - K^2 - 4K - 6$.
36. $n^3 - 7n + n^2 - 3$.
37. $D^3 + 15 - 17D + D^2$.
38. $432 + 2F^3$.
39. $1 - 2x + x^2 + 2ab - b^2 - a^2$.
40. $P^4 + P^2 + 1$.
41. $ax^2 + (ab + 1)x + b$.
42. $2t^2 - (2a + b)t + ab$.
43. $2H^2 + (4a + b)H + 2ab$.
44. $az^2 + (b - a)z - b$.
45. $x^4y^4 + 9ax^2y^2 + 14a^2$.
46. $220 - 2V - 2V^2$.
47. $20c^3 - 60c^2 + 45c$.
48. $1 + 32Q^5n^{10}$.
49. $1 - d^9$.
50. $y^3 - 11y^2 + 31y - 21$.
51. $L^3 - 6L^2 + 11L - 6$.
52. $y^2 - 9z^2 + 25x^2 - 10xy$.

53. $3(a-b)^2-14(a-b)+8$.

54. $7(a+b)^2-11c(a+b)-6c^2$.

55. $x^4-x^3-17x^2+5x+60$.

56. a^3-a^2A-a+A.

57. $2m^3b+3m^2b-8mb-12b$.

58. $a^{10}+ab^9$.

59. $v^{16}-t^{16}$.

60. $P^4-P^2+12PQ-36Q^2$.

61. $1-n+n^6-n^7$.

62. $289g^7-83g^5+g^3$.

63. x^3+x+y^3+y.

64. $M-N+M^3-N^3$.

65. $1-a^{21}$.

66. $x^2+4x+4-4a^2+4ay-y^2$.

28. Highest Common Factor. — Since the *highest common factor* of two or more expressions must contain all factors common to the expressions, and no others,

To find the highest common factor of two or more expressions, find the prime factors of each. Then form the product of all prime factors that are common to the expressions, using each factor the least number of times that it occurs in any one of the expressions.

EXERCISES

Find the highest common factor of:

1. 24, 42, 54, and 78.
2. 36, 84, 120, and 108.
3. 216, 360, 504, and 576.
4. $12a^3b^4$, $18a^4b^3$, and $24a^3b^3c$.
5. $16m^2na^4$, $42ma^3$, and $18m^3a^2b$.
6. x^2-1, x^2-3x+2, and x^2+x-2.
7. V^3-1, V^2-2V+1, and V^2-3V+2.
8. a^5+b^5, a^3+b^3, and a^2-b^2.
9. x^5-xy^2, and $x^3+x^2y+xy+y^2$.
10. $7N^3-2N^2-5$ and $7N^3+12N^2+10N+5$.
11. $p^2+3pq+2q^2$, $p^2+5pq+4q^2$, and $p^2-6pq-7q^2$.
12. $1-x^6$, $1-2x^3+x^6$, and $1+x+x^2$.
13. $R^2-7R+10$, and $4R^3-25R^2+20R+25$.
14. $6v^2t+4vt^2-2t^3$, and $8v^3+4v^2t-4vt^2$.
15. $4P^2+20P$, P^3+4P^2-5P, and $4aP^2+20aP$.
16. m^2+m-6, m^3-2m^2-m+2, m^3+3m^2-6m-8.

29. Lowest Common Multiple. — The lowest common multiple of two or more expressions must contain all factors of the expressions, and no others. Hence,

To find the lowest common multiple of two or more expressions, find the prime factors of each. Then form the product of all of the different kinds of factors, using each kind of factor the greatest number of times that it occurs in any one of the expressions.

EXERCISES

Find the lowest common multiple of:

1. 18, 24, 42, and 54.
2. 30, 45, 75, and 120.
3. 32, 56, 63, and 112.
4. $36\,a^{10}b$, $9\,a^2b^5$, and $16\,a^3b^2$.
5. $25\,\pi R^2(R-r)(2\,R+r)^2$, and $45\,\pi Rr(2\,R+r)^3$.
6. x^2-1, and $x^2-2\,x+1$.
7. x^3-y^3, x^3-xy^2, and $2\,x^2y$.
8. N^4-1, $N^2+3\,N+2$, and N^2+N-2.
9. m^2-n^2, m^3-n^3, and m^4-n^4.
10. $1-V^2$, $1-V^4$, and $1-V^8$.
11. $a^3+2\,a^2-3\,a$, $2\,a^3+6\,a^2+2\,a+6$, and $3\,a^3-3\,a^2$.
12. P^4-Q^4, $P^4+2\,P^2Q^2+Q^4$, and $P^4-2\,P^2Q^2+Q^4$.
13. w^2-v^2, $w^3+w^2v-wv^2-v^3$, and $w^3-w^2v-wv^2+v^3$.
14. $y^4-13\,y^2+36$, and $y^4-y^3-7\,y^2+y+6$.
15. $27\,T^3-8$, $9\,T^2-4$, and $9\,T^2-12\,T+4$.
16. $6\,M^3-7\,M^2N-2\,MN^2$, and $3\,M^2+MN-4\,N^2$.

SUPPLEMENTARY EXERCISES

Factor:

1. $1-x-x^2+x^3$.
2. $4(a-b)^3-a+b$.
3. $4\,c^3-c^2-8\,c+2$.
4. $x^{16}-y^{16}$.
5. $2^{5x}-32$.
6. x^3y^3-512.

7. $a^{3n} - a^{2n}b^{n}$.
8. $x^{2n} - x^{n} - 2$.
9. $t^{4n} + 3\,t^{2n}v^{n} - 28\,v^{2n}$.
10. $p^{2n} - q^{2n}$.
11. $A^{4n} - B^{4n}$.
12. $x^{2n+1} - xy^{n}$.
13. $V^{3t} - 1$.
14. $nk^{5n} - n^{6n}$.
15. $a^{2n} - b^{2n} - a^{n}c^{n} + b^{n}c^{n}$.
16. $6\,b^{2n} + b^{n}c^{n} - 15\,c^{2n}$.
17. $x^{2n-4} - y^{2n+4}$.
18. $\pi R^{2n} + (\pi r + 1)R^{n} + r$.
19. $V^{4t} + V^{2t} + 1$.
20. $N^{5a} + 1$.
21. $16^{n} + 2 \cdot 2^{2n} + 1$.
22. $a^{4} - 11\,a^{2}b^{2} + b^{4}$.
23. $4^{n} + 3 \cdot 2^{n} + 2$.
24. $a^{15} + b^{15}$.
25. $9^{n} - 4 \cdot 3^{n} - 5$.
26. $4^{a} - 16^{b}$.
27. $8^{a} - 27^{b}$.
28. $2 \cdot 3^{2} - 9\,a - 2\,a^{2}$.

Find the highest common factor of:

29. $D^{2m} - 1$, and $D^{3m} - 1$.
30. $a^{5n} - a^{n}b^{2n}$, and $a^{3n} + a^{2n}b^{n} + a^{n}b^{n} + b^{2n}$.
31. $2\,V^{2t} + V^{t} - 6$, and $6\,V^{2t} - 7\,V^{t} - 3$.
32. $x^{3} + 3\,x^{2} - 7\,x - 21$, and $x^{3} + 3\,x^{2} - 3\,x - 9$.
33. $4\,ax - 6\,x - 2\,ay + 3\,y$, and $4\,a^{2} - 10\,a + 6$.
34. $2\,a^{2} - 15\,a + 25$, and $a^{3} - 5\,a^{2} - a + 5$.

Find the lowest common multiple of:

35. $x^{3} - y^{3}$, $x^{3} + y^{3}$, $x^{6} - y^{6}$, and $x^{12} - y^{12}$.
36. $x^{2} - 4$, $x^{2} - 7\,x + 10$, and $x^{3} - 5\,x^{2} + 4x - 20$.
37. $9\,by^{3} - 3\,by^{2} - 42\,by$, and $6\,b^{2} - 26\,b + 28$.
38. $a^{2n} - b^{2n}$, $a^{3n} - b^{3n}$, and $a^{4n} - b^{4n}$.
39. $4\,P^{2x} - 4\,P^{x} + 1$, $4\,P^{x} - 1$, and $4\,P^{2x} + 4\,P^{x} + 1$.
40. $a^{n+1}x - a^{n+1}y - b^{n-1}x + b^{n-1}y$, and $x^{2} - 2\,xy + y^{2}$.

CHAPTER III

FRACTIONS

30. Fractions. — A **fraction** is an indicated quotient of two number expressions. The dividend is called the **numerator** and the divisor the **denominator** of the fraction. The numerator and denominator are called the **terms** of the fraction.

31. The Signs of a Fraction. — In an algebraic fraction three signs are involved: (1) the sign before the fraction; (2) the sign of the numerator; and (3) the sign of the denominator.

Since a fraction is an indicated quotient, by the *laws of signs in division*, $\frac{+a}{+b}$ and $\frac{-a}{-b}$ are both positive and are equal, and $\frac{-a}{+b}$ and $\frac{+a}{-b}$ are both negative and are equal. Hence,

$$\frac{+a}{+b}=\frac{-a}{-b}=-\frac{-a}{+b}=-\frac{+a}{-b}.$$

It follows that to preserve the value of a fraction:

(1) *If the signs of both terms are changed, the sign before the fraction must be left unchanged.*

(2) *If the sign of only one term is changed, the sign before the fraction must be changed.*

If the numerator is a polynomial, its sign is changed by changing the sign of each of its terms.

If the numerator or denominator is the product of two or more factors, its sign is changed by changing the signs of an odd number of factors, and it is left unchanged by changing the signs of an even number of factors.

Thus, in $\frac{1-x}{(x-3)(x-4)}$ the denominator $(x-3)(x-4)=x^2-7x+12$. If the sign of $x-3$ is changed, by changing the signs of its terms, we get

$(3-x)(x-4) = -x^2+7x-12$. If the signs of both factors are changed, we get $(3-x)(4-x) = x^2-7x+12$.

EXERCISES

Write with positive numerators and denominators:

1. $\frac{-4}{7}$.
2. $\frac{6}{-9}$.
3. $\frac{-2}{-5}$.
4. $\frac{+8}{-9}$.
5. $\frac{-2x^2}{-3y^2}$.
6. $\frac{4ab}{-5c^3}$.
7. $\frac{-1}{t^2}$.
8. $\frac{-\pi r}{-R}$.

Write with denominators $m-n$:

9. $\frac{1}{n-m}$.
10. $\frac{-mn}{n-m}$.
11. $\frac{a-b}{n-m}$.

Write all of the fractions with the same denominator:

12. $\frac{3}{a-b}-\frac{4}{b-a}$.
13. $\frac{x}{x-1}+\frac{1}{1-x}-\frac{x+1}{1-x}$.
14. $\frac{1+2m}{m^2-1}-\frac{3m}{1-m^2}+\frac{2-m}{1-m^2}$.
15. $\frac{6\pi r}{r^2-s^2}-\frac{r^2}{s^2-r^2}+\frac{r^2+s^2}{r^2-s^2}$.

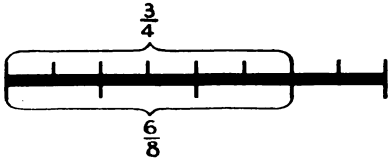
Write with the factors of denominators in descending powers of a, the first term of each factor being positive:

16. $\frac{1}{(a-b)(c-a)}$.
17. $\frac{a^3}{(a-b)(c-a)(d-a)}$.
18. $\frac{a+b+c+d}{(b-a)(c-a)(d-a)}$.

32. Reduction of a Fraction to an Equivalent Fraction. — *Multiplying or dividing both terms of a fraction by the same expression does not change the value of the fraction. That is,*

$$\frac{a}{b}=\frac{na}{nb} \text{ and } \frac{a}{b}=\frac{a\div n}{b\div n}.$$

This principle may be illustrated graphically as follows:



If each of the 3 fourths in $\frac{3}{4}$ is halved, they make 6 eighths. That is, $\frac{3}{4}=\frac{6}{8}=\frac{2\times 3}{2\times 4}$. Starting with $\frac{6}{8}$, if each of the 6 eighths in $\frac{6}{8}$ is doubled, they make 3 fourths. That is, $\frac{6}{8}=\frac{3}{4}=\frac{6\div 2}{8\div 2}$.

Multiplying or dividing both terms of a fraction by the same expression is called **reducing** the fraction.

33. Reduction of Fractions to Lowest Terms. — *To reduce a fraction to its lowest terms, factor the numerator and denominator, and divide both terms by their common factors.*

The division may be indicated by *cancellation* of the common factors of the terms.

Thus,
$$\frac{a^3-ab^2}{a^4-ab^3}=\frac{\cancel{a}\cancel{(a-b)}(a+b)}{\cancel{a}\cancel{(a-b)}(a^2+ab+b^2)}=\frac{a+b}{a^2+ab+b^2}.$$

EXERCISES

Reduce to lowest terms:

1. $\frac{18}{24}$.
2. $\frac{15}{45}$.
3. $\frac{42}{49}$.
4. $\frac{64}{72}$.
5. $\frac{250}{350}$.
6. $\frac{45}{630}$.
7. $\frac{xy^2}{yz^2}$.
8. $\frac{4A}{8A^3}$.
9. $\frac{6a^3b^2}{9a^2b^3}$.
10. $\frac{2\pi R}{4\pi R^3}$.
11. $\frac{x^2-1}{x^3-1}$.
12. $\frac{a^3+b^3}{a^3+a^2b}$.
13. $\frac{m^4-n^4}{m^4+2m^2n^2+n^4}$.
14. $\frac{t^2-5t-14}{t^2+8t+12}$.
15. $\frac{10p^2-13pq-3q^2}{8p^2-2pq-15q^2}$.
16. $\frac{1+N^3}{N^3-4N^2-5N}$.
17. $\frac{w^3v-wv^3}{w^3v^2+w^2v^3}$.
18. $\frac{F^5+1}{(F+1)^5}$.
19. $\frac{A^2+AB+B^2}{A^4+A^2B^2+B^4}$.
20. $\frac{a^2+an+ax+nx}{a^2+an-ay-ny}$.

34. Reduction of Fractions to a Common Denominator. — Two or more fractions may be reduced to equivalent fractions that have the same denominator, called their **common denominator**. The

lowest common denominator of two or more fractions is the lowest common multiple of their denominators.

To reduce two or more fractions to equivalent fractions with a common denominator, factor all denominators, and form their lowest common multiple for the common denominator. Multiply both terms of each fraction by the factor or factors of the common denominator that its denominator does not already contain.

EXERCISES

Reduce to equivalent fractions with the lowest common denominator:

1. $\frac{5}{6}, \frac{3}{4}, \frac{1}{2}$.
2. $\frac{1}{3}, \frac{1}{4}, \frac{3}{8}$.
3. $\frac{2}{7}, \frac{1}{2}, \frac{2}{3}$.
4. $\frac{5}{12}, \frac{3}{16}, \frac{7}{24}$.
5. $\frac{2}{15}, \frac{3}{25}, \frac{7}{45}$.
6. $\frac{1}{8}, \frac{5}{36}, \frac{20}{27}$.
7. $\frac{1}{2}, \frac{16}{21}, \frac{25}{28}$.
8. $\frac{9}{16}, \frac{2}{9}, \frac{4}{15}$.
9. $\dfrac{4}{3a^2}, \dfrac{1}{5a}, \dfrac{7}{12a^3}$.
10. $\dfrac{a}{b^2}, \dfrac{b}{c^2}, \dfrac{c}{a^2}$.
11. $\dfrac{m}{np}, \dfrac{p}{mn}, \dfrac{n}{mp}$.
12. $\dfrac{1}{1-x}, \dfrac{1}{1+x}, \dfrac{x}{1-x^2}$.
13. $\dfrac{a}{a+b}, \dfrac{b}{a-b}, \dfrac{ab}{a^2+b^2}$.
14. $\dfrac{M}{N}, \dfrac{M^2}{MN-N^2}, \dfrac{MN}{M^2-N^2}$.
15. $\dfrac{t}{t^2-9}, \dfrac{t-1}{t^2-5t+6}$.
16. $\dfrac{R-3}{R^2-9R+14}, \dfrac{-3}{R^2-4}$.
17. $\dfrac{m+n}{m^3-n^3}, \dfrac{2n}{m^2n-mn^2}$.
18. $\dfrac{3}{x^2+3x+2}, \dfrac{5}{2x^2+5x+2}, \dfrac{7}{2x^2+3x+1}$.
19. $\dfrac{1}{(a-b)(a-c)}, \dfrac{1}{(b-c)(b-a)}, \dfrac{1}{(c-a)(c-b)}$.
20. $\dfrac{y+z}{(y-x)(z-x)}, \dfrac{z+x}{(y-z)(y-x)}, \dfrac{x+y}{(z-x)(z-y)}$.

35. Addition and Subtraction of Fractions. — *To add or subtract fractions, reduce them to their lowest common denominator, then find the sum or difference of their numerators, and place the result over the common denominator.*

Before beginning to factor the denominators, it is best to rewrite the fractions, if necessary, so that the denominators are all arranged in descending or all in ascending powers of some letter of arrangement, with the first term of each denominator positive.

EXAMPLE. — Simplify $\frac{n}{1+n}-\frac{n}{1-n}+\frac{n^2}{n^2-1}$.

Rewriting the first two fractions, with denominators in descending powers of n, making changes of sign as in § 31, the expression becomes

$$\begin{aligned}\frac{n}{n+1}+\frac{n}{n-1}+\frac{n^2}{n^2-1}&=\frac{n}{n+1}+\frac{n}{n-1}+\frac{n^2}{(n+1)(n-1)}\\&=\frac{n^2-n}{(n+1)(n-1)}+\frac{n^2+n}{(n+1)(n-1)}+\frac{n^2}{(n+1)(n-1)}\\&=\frac{3n^2}{(n+1)(n-1)}\text{ or }\frac{3n^2}{n^2-1}.\end{aligned}$$

36. Reduction of Mixed Expressions to Fractions. — An expression that contains no fractions with literal denominators is an **integral expression**; one that consists of one or more fractions is a **fractional expression**; and one consisting of an integral part and a fractional part is a **mixed expression.**

A mixed expression may be changed to a fraction by the process of § 35, the integral part being written in the form of a fraction with the denominator 1.

EXAMPLE. — Reduce $m-n+\frac{2mn}{m-n}$ to a fraction.

$$\begin{aligned}m-n+\frac{2mn}{m-n}&=\frac{m-n}{1}+\frac{2mn}{m-n}\\&=\frac{m^2-2mn+n^2}{m-n}+\frac{2mn}{m-n}\\&=\frac{m^2+n^2}{m-n}.\end{aligned}$$

EXERCISES

Simplify by adding or subtracting, as indicated:

1. $\frac{4}{5}+\frac{3}{4}$. 2. $\frac{7}{16}-\frac{1}{12}$. 3. $\frac{5}{12}+\frac{2}{7}-\frac{1}{8}$. 4. $\frac{15}{16}-\frac{2}{5}+\frac{8}{15}$.

5. $\frac{3}{a^2}+\frac{1}{2a}$. 7. $\frac{a}{bc}+\frac{b}{ac}+\frac{c}{ab}$. 9. $\frac{t}{1-t^2}-\frac{1}{t-1}$.

6. $\frac{1}{x}+\frac{x-1}{x^2}-\frac{3}{2x}$. 8. $\frac{N+1}{N-1}-\frac{N-1}{N+1}$. 10. $\frac{5}{s^2-s}+\frac{2}{s^2+s}$.

11. $\dfrac{r-2}{r^2+10r+16}+\dfrac{r+1}{r^2+9r+14}.$ 13. $\dfrac{V-3}{V+3}+\dfrac{V+3}{V-3}+\dfrac{V^2}{9-V^2}.$

12. $\dfrac{2\pi R}{\pi R+\pi H}-\dfrac{2\pi R}{\pi R-\pi H}.$ 14. $\dfrac{a^2}{a^3-b^3}-\dfrac{1}{b-a}.$

15. $\dfrac{1}{ab-ac-b^2+bc}+\dfrac{1}{bc-ab-c^2+ac}.$

16. $\dfrac{1}{(a-b)(b-c)}+\dfrac{1}{(b-a)(a-c)}-\dfrac{1}{(c-a)(c-b)}.$

17. $\dfrac{M^2}{(M-N)(M-P)}-\dfrac{N^2}{(P-N)(N-M)}+\dfrac{P^2}{(P-M)(N-M)}.$

18. $\dfrac{1}{R+r}-\dfrac{1}{R-r}+\dfrac{1}{R+3r}-\dfrac{1}{R-3r}.$

Suggestion. — First combine the first two fractions, then the last two, then add the results. This will save much multiplication.

19. $\dfrac{1}{n-4}+\dfrac{1}{n-5}-\dfrac{1}{n+4}-\dfrac{1}{n+5}.$

20. $\dfrac{2}{a-b}-\dfrac{4}{c+d}-\dfrac{2}{a+b}+\dfrac{4}{c-d}.$

21. $\dfrac{1}{x^2-9}-\dfrac{1}{x^2+9}+\dfrac{9}{x^2-1}-\dfrac{9}{x^2+1}.$

Reduce to a fraction:

22. $t-\dfrac{1}{t}.$ 24. $\dfrac{a^2}{a-b}-a.$ 26. $m+n-\dfrac{2mn}{m+n}.$

23. $\dfrac{V}{V+3}+2.$ 25. $1-R+\dfrac{R^2}{1+R}.$ 27. $y+2+\dfrac{3-y^2}{y-4}.$

28. $1+\dfrac{M}{N}+\dfrac{M^2}{N^2}.$ 30. $\dfrac{a}{a-b}-\dfrac{b}{a-b}-1.$

29. $x^2+xy+y^2+\dfrac{y^3}{x-y}.$ 31. $D^2-\dfrac{2D^2-D^4}{D^2-D+1}+D+1.$

37. Multiplication of Fractions. — As in arithmetical fractions, *The product of two or more fractions is the fraction whose numerator is the product of their numerators and whose denominator is the product of their denominators.*

In the multiplication of fractions, if a factor is common to a numerator and any denominator, the product may be reduced to lower terms by canceling this factor from the terms of the product. In such a case, work may be saved by factoring the *terms* of the fractions and canceling any factor common to any numerator and any denominator, before performing the multiplications.

Thus, $$\frac{x^2-1}{x^2+5x+6}\times\frac{x^2+6x+8}{x^2+6x-7}=\frac{\cancel{(x-1)}(x+1)}{(x+3)\cancel{(x+2)}}\times\frac{\cancel{(x+2)}(x+4)}{\cancel{(x-1)}(x+7)}$$
$$=\frac{(x+1)(x+4)}{(x+3)(x+7)}\text{ or }\frac{x^2+5x+4}{x^2+10x+21}.$$

Integral, mixed, or fractional expressions to be multiplied should first be reduced to fractions.

Thus, $$(m^2-n^2)\left(1-\frac{m}{m+n}\right)=\frac{\cancel{(m+n)}(m-n)}{1}\times\frac{n}{\cancel{m+n}}$$
$$=mn-n^2.$$

EXERCISES

Find the product of:

1. $\frac{3}{4}\times\frac{2}{5}$.
2. $\frac{2}{3}\times\frac{6}{7}$.
3. $6\times\frac{2}{3}$.
4. $10\times\frac{4}{5}$.
5. $\frac{1}{2}\times\frac{4}{9}\times\frac{3}{8}$.
6. $1\frac{5}{12}\times\frac{10}{16}\times\frac{9}{15}$.
7. $9\frac{1}{2}\times\frac{5}{12}\times\frac{2}{5}$.
8. $21\times\frac{25}{36}\times\frac{4}{35}$.
9. $\frac{a^2b}{cd}\times\frac{ac^2}{b^2d}\times\frac{d^3}{a^2c}$.
10. $\frac{x^2}{y^2}\times\frac{y^3z}{x^3}\times\frac{x}{yz}$.
11. $\frac{m^4}{n^2p}\times\frac{mn^2}{p^2}\times\frac{np^3}{m^2}$.
12. $\frac{x^4-1}{x^2+x+1}\times\frac{x^3-1}{x^2-1}$.
13. $\frac{v-t}{v+2t}\times\frac{v^2-2t^2}{v^3-t^3}$.
14. $\frac{n^2-14n-15}{n^2-4n-45}\times\frac{n^2-6n-27}{n^2-12n-45}$.
15. $\frac{\pi R^2+\pi RH}{R-H}\times\frac{H}{\pi R^2-\pi H^2}$.
16. $\frac{b^3-1}{b+1}\times\frac{b^2-1}{b^2+b+1}$.
17. $\left(1-\frac{27T^3}{V^3}\right)\left(\frac{1}{V-3T}\right)$.
18. $(A^2+A+1)\left(\frac{1}{A^3-1}\right)$.
19. $\left(\frac{a^2+ab+b^2}{a^3-b^3}\right)(a^2-b^2)$.
20. $\left(\frac{1}{N^2}-1\right)\left(1+\frac{1}{N^2}\right)$.
21. $\left(1+\frac{1}{F}+\frac{1}{F^2}\right)\left(\frac{1}{F^3-1}\right)$.

22. $\left(1-\frac{b^2}{a^2}\right)\left(\frac{1}{a-b}\right)$.

23. $\left(\frac{mn}{m^2+n^2}\right)\left(1-\frac{m^4}{n^4}\right)$.

24. $\left(\frac{h^2+hk}{h^2+k^2}\right)\left(\frac{h}{h-k}-\frac{k}{h+k}\right)$.

25. $\left(\frac{1}{P}-1\right)\left(\frac{P}{1+P}-\frac{1-P}{P}\right)$.

Multiply by the lowest common denominator:

26. $\frac{1}{n-1}+\frac{1}{n+1}$.

27. $\frac{A}{A^2-B^2}-\frac{1}{A+B}$.

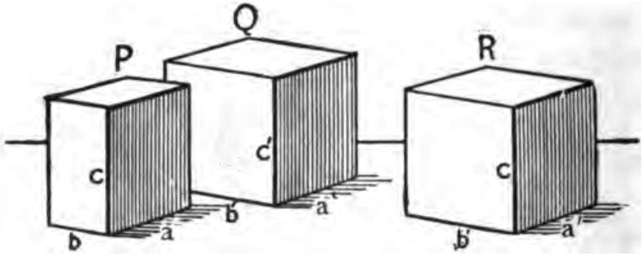
28. $\frac{1}{x-y}-\frac{xy+2y^2}{x^3-y^3}$.

29. $\frac{2K^2}{K^4-1}+\frac{1}{K^2+1}-\frac{1}{K^2-1}$.

30. $\frac{2v+1}{v^3-1}-\frac{3}{v^2-1}$.

31. $4-\frac{7M-2}{M^2-M}$.

32. P, Q, and R are the volumes of the three rectangular solids shown in the figure. It is known from geometry that $\frac{P}{R}=\frac{ab}{a'b'}$ and that $\frac{R}{Q}=\frac{c}{c'}$.



Show by multiplication that $\frac{P}{Q}=\frac{abc}{a'b'c'}$.

38. Division of Fractions. — *To find the quotient of two fractions multiply the dividend by the divisor inverted.*

That is, $\frac{a}{b}\div\frac{c}{d}=\frac{a}{b}\times\frac{d}{c}$.

For, the product of the quotient $\frac{a}{b}\times\frac{d}{c}$ and the divisor $\frac{c}{d}$ equals the dividend $\frac{a}{b}$, by cancellation.

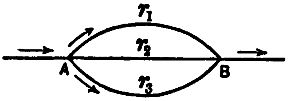
If the dividend or divisor is an integral, mixed, or fractional expression, it must be changed to a fraction, and the division performed by the above rule.

Example.—$\left(1-\frac{x^2}{y^2}\right)\div\frac{x+y}{xy}=\frac{y^2-x^2}{y^2}\times\frac{xy}{x+y}=\frac{xy-x^2}{y}.$

EXERCISES

Perform the indicated divisions:

1. $\frac{3}{4}\div\frac{6}{7}$.
2. $\frac{15}{16}\div\frac{9}{10}$.
3. $12\div\frac{3}{5}$.
4. $\frac{3}{8}\div 4$.
5. $21\div\frac{7}{9}$.
6. $3\frac{2}{3}\div\frac{5}{6}$.
7. $\frac{16}{27}\div 3\frac{1}{3}$.
8. $1\frac{5}{6}\div 6\frac{7}{8}$.
9. $\frac{4m^2n}{5pq^2}\div\frac{6n^2}{q^2}$.
10. $\frac{4\pi R^2}{9D}\div\frac{2\pi R}{3D}$.
11. $\frac{x^2-1}{x}\div\frac{x+1}{x^2}$.
12. $\frac{9-V^2}{2+V}\div\frac{6-2V}{4-V^2}$.
13. $\frac{a^3+b^3}{2ab}\div\frac{a^2+ab}{2b}$.
14. $\frac{t^2-1}{t}\div(1+t)$.
15. $\frac{m^3-n^3}{6m^2+mn-n^2}\div\frac{4m^2+mn-5n^2}{6m^2-5mn-4n^2}$.
16. $\left(1-\frac{P^3}{Q^3}\right)\div\frac{P-Q}{PQ}$.
17. $(a^2+ab+b^2)\div\frac{a^4-ab^3}{b}$.
18. $\left(x+1+\frac{3}{x-1}\right)\div\left(x+2-\frac{1}{x-1}\right)$.
19. $\left(\frac{M^2+N^2}{N}-M\right)\div\left(\frac{1}{N}-\frac{1}{M}\right)$.
20. $\left(\frac{k^2}{t^2}-\frac{2k}{t}+1\right)\div\left(1-\frac{k^3}{t^3}\right)$.



21. If a conductor of an electrical current is divided between the points A and B into three branches with resistances r_1, r_2, r_3, respectively, the total resistance of the conductor between A and B is $1\div\left(\frac{1}{r_1}+\frac{1}{r_2}+\frac{1}{r_3}\right)$. Simplify this expression.

39. Complex Fractions.—Fractions one or both of whose terms contain fractions are called **complex fractions.** A complex fraction may be simplified by reducing it to a simple fraction or an integral expression. Since a complex fraction is an indicated quotient,

To simplify a complex fraction, reduce both terms to simple fractions, then perform the indicated division.

A complex fraction of the special form indicated in the example below is called a **continued fraction.** To simplify such a fraction, begin with the last or lowest complex fraction, and simplify step by step.

EXAMPLE. — Simplify $\dfrac{1}{x+\dfrac{1}{x-\dfrac{1}{x}}}$.

Simplifying $\dfrac{1}{x-\dfrac{1}{x}}$, $\quad \dfrac{1}{x+\dfrac{1}{x-\dfrac{1}{x}}} = \dfrac{1}{x+\dfrac{x}{x^2-1}} = \dfrac{x^2-1}{x^3}$.

EXERCISES

Simplify:

1. $\dfrac{\frac{7}{8}}{\frac{9}{16}}$.

2. $\dfrac{8}{\frac{5}{6}}$.

3. $\dfrac{3\frac{1}{3}}{2\frac{1}{2}}$.

4. $\dfrac{\dfrac{M}{N}}{\dfrac{M^3}{N^3}}$.

5. $\dfrac{\dfrac{x^3}{y^2}}{x}$.

6. $\dfrac{1-x}{\dfrac{1}{x^2}-1}$.

7. $\dfrac{\dfrac{1}{A+B}}{\dfrac{1}{A-B}}$.

8. $\dfrac{1-\dfrac{m}{n}}{1+\dfrac{m}{n}}$.

9. $\dfrac{\dfrac{b}{c}-\dfrac{c}{b}}{1+\dfrac{c}{b}}$.

10. $\dfrac{\dfrac{a}{a+b}-1}{a+b}$.

11. $\dfrac{2t+\dfrac{t^2+s^2}{s}}{\dfrac{t}{s}+1}$.

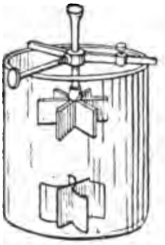
12 $\dfrac{\dfrac{a^6+b^6}{a^6-b^6}}{\dfrac{a^4-a^2b^2+b^4}{a^4+a^2b^2+b^4}}$.

13. $\dfrac{\dfrac{m}{n}+1+\dfrac{n}{m}}{\dfrac{m^2}{n^2}+1+\dfrac{n^2}{m^2}}$.

14. $\dfrac{1}{x+\dfrac{1}{1+\dfrac{x+1}{1-x}}}$.

15 $\dfrac{1}{a-\dfrac{a^2-1}{a+\dfrac{1}{a-1}}}$.

16. $\dfrac{2}{2+\dfrac{2}{2+\dfrac{2}{2+\dfrac{2}{x}}}}$.



17. If n equal electric cells are connected by wires in one way (parallel), so as to form a battery, the current strength produced by the battery is $\dfrac{E}{R+\dfrac{r}{n}}$. Simplify this fraction.

18. It has been shown that π, the ratio of the circumference to the diameter of a circle, is equal to the continued fraction in the margin. Using only so much of the fraction as is here written, find the value of π approximated to hundredths.

$$\cfrac{4}{1+\cfrac{1}{2+\cfrac{9}{2+\cfrac{25}{2+\cfrac{49}{2+\cfrac{81}{2+\text{etc.}}}}}}}$$

SUPPLEMENTARY EXERCISES

Simplify:

1. $\dfrac{n^2-100}{n^2-n-90}\div\dfrac{n^2+10n}{n^3+9n^2}\div\dfrac{2n^2t+3nt}{4n+6}$.

2. $\left(1-V+\dfrac{2V^2}{1+V}\right)\div\left(\dfrac{1}{1+V}-\dfrac{1}{1-V}\right)\times\left(\dfrac{3V^2}{V^4-1}\right)$.

3. $\left(\dfrac{A}{A-B}-\dfrac{B}{A+B}\right)\left(\dfrac{1}{A^2}-\dfrac{1}{B^2}\right)\div\left(\dfrac{1}{A^2}+\dfrac{1}{B^2}\right)$.

4. $\left(\dfrac{x^2-y^2-z^2-2yz}{x^2-xy-xz}\right)\div\left(1-\dfrac{2z}{x+y+z}\right)$.

5. $\left(\dfrac{5a^2}{b}+b\right)\left(\dfrac{b^2}{5a^2}-1\right)\div\dfrac{b}{a}+\dfrac{a+b}{b}$.

6. $\left(\dfrac{m}{n}-\dfrac{n}{m}\right)\div\left(\dfrac{n}{m}-\dfrac{m}{n}\right)+\dfrac{m^2n^2-n^4}{m+n}\times\dfrac{1}{mn^2-n^3}$.

7. $\left(\frac{a+b}{b+c}-\frac{a+b}{a+c}\right)\left(\frac{a}{a+b}+\frac{c}{a-b}-\frac{a^2-2ab-c^2}{a^2-b^2}\right).$

8. $\left(\frac{2a-2n}{a+n}-\frac{a+n}{a-n}\right)\left(\frac{1}{a}+\frac{1}{n}\right)-\left(\frac{1}{a}-\frac{1}{n}\right)\div\left(\frac{1}{a}+\frac{1}{n}\right).$

9. $\left(\frac{w^2+v^2}{w^2-v^2}-\frac{w-v}{w+v}\right)\left(\frac{w+v}{2wv}\right)+\frac{w^2-1-v^2}{w-v}.$

10. $\frac{a^3-x^3}{a^3+x^3}\times\frac{a^2-x^2}{a^2+x^2}\times\frac{a-x}{a+x}\times\frac{a^2-ax+x^2}{a^2+ax+x^2}\times\frac{a^2+2ax+x^2}{a^2-2ax+x^2}.$

11. $\frac{x+1}{y}\times\frac{y^2+2y+1}{x^2-1}\times\frac{(x-1)y}{(y+1)^2}.$

12. $\frac{a+b}{m}\div\frac{b^2-a^2}{m^2}\times\left(\frac{1}{x}\div\frac{m}{a-b}\right).$

13. $\left(1+\frac{1}{a}\right)\div\left(a-\frac{1}{a}\right)\times\left(1-\frac{1}{a}\right).$

14. $\dfrac{a-\dfrac{a-x}{1+ax}}{1+\dfrac{a(a-x)}{1+ax}}.$

15. $\dfrac{\dfrac{m^2+mn+n^2}{m^3+n^3}}{\dfrac{m^3-n^3}{m^2-mn+n^2}}.$

16. $\dfrac{\dfrac{a^6+b^6}{a^3+b^3}}{\dfrac{a^6-b^6}{a^3-b^3}}.$

17. $\dfrac{\dfrac{a^2+b^2}{a^2-b^2}-\dfrac{a^2-b^2}{a^2+b^2}}{\dfrac{a+b}{a-b}-\dfrac{a-b}{a+b}}.$

18. $\dfrac{\dfrac{AB-B^2}{A^2-AB}+\dfrac{A^2-B^2}{AB}}{\dfrac{A(A+B)}{B(A-B)}}\div\dfrac{1}{\dfrac{(A+B)^2}{A^2-B^2}}.$

CHAPTER IV

LINEAR EQUATIONS IN ONE UNKNOWN NUMBER

40. The Equation. — An **equation** is the statement that two expressions are equal in value.

Thus, $a^2 - b^2 = (a+b)(a-b)$, and $2N + 4 = 5N + 1$ are equations.

The two expressions connected by the sign = are called the **members** of the equation.

If an equation is true for all particular values of the literal numbers involved, it is called an **identical equation**, or simply an **identity**. In the work of the preceding chapters all of the processes really consisted of establishing identities.

Thus, $a^2 - b^2 = (a+b)(a-b)$ is true when $a = 2$ and $b = 1$, when $a = 5$ and $b = 2$, and for any values whatsoever that may be assigned to a and b, and hence is an identity. This identity was established in *factoring.*

If an equation is not true for all particular values of the literal numbers involved, it is called a **conditional equation**, or simply an **equation.**

Thus, $5t - 4 = 2t + 8$ is true only when t is 4.

It is by use of conditional equations that problems are solved. The unknown numbers, whose values are sought, are represented by letters, and the relations between the unknown numbers and the known numbers of the problems are expressed by means of equations. An equation is **solved** when the values of the unknown number are found that reduce the equation to an **identity.**

41. Integral and Fractional Equations. — If no unknown number of an equation be involved in the denominator of a fraction, the equation is **integral.** If one or more unknown numbers of an equation be involved in the denominators of one or more fractions, the equation is **fractional.**

Thus, $6A^2 - 4 = 7A$ is *integral*, and $\frac{1}{x} - \frac{1}{y} = 8$ is *fractional.*

42. The Degree of an Equation. — The **degree of an integral equation** is the greatest sum of the exponents of its unknown numbers in any one term. If an equation has only one unknown number, its degree is the largest exponent of the unknown number in the equation.

Thus, $V + 5 = 9 - 2V$ is of the *first degree.*
$x^2 + 7x - 2 = 0$ is of the *second degree.*
$x^3 - xy^2 = 16x + 2$ is of the *third degree.*
$m^2n^2 + 2mn = 0$ is of the *fourth degree.*

An equation of the first degree is called a **linear equation.**

One of the second degree is called a **quadratic equation.**

One of the third degree is called a **cubic equation.**

43. Solving an Equation in One Unknown Number. — If a conditional equation contains only one unknown number, the values of the unknown number for which the equation is true are called the **roots** of the equation, and are said to **satisfy** the equation.

The process of finding the values of the roots of an equation is called **solving the equation.**

As shown in the First Course, equations may be solved by applying one or more of the following principles, called **axioms**:

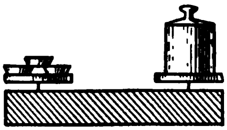
I. *If equal numbers are added to equal numbers, the sums are equal.*

II. *If equal numbers are subtracted from equal numbers, the remainders are equal.*

III. *If equal numbers are multiplied by equal numbers, the products are equal.*

IV. *If equal numbers are divided by equal numbers (not zero), the quotients are equal.*

The use of these axioms in solving an equation may be compared with changing the weights on the two pans of a balance. Just as adding equal weights to both pans, removing equal weights from both pans, etc., when the instrument is balanced will keep it balanced, so adding equal expressions to, or subtracting equal expressions from, both members of an equation, etc., will preserve the equality.



Example. — Solve $6E - 5 = 4E + 1$.

Adding 5 to each member, $6E = 4E + 6$ Ax. I.

Subtracting $4E$ from each member, $2E = 6$. Ax. II.

Dividing each member by 2, $E = 3$. Ax. IV.

EXERCISES

Solve by use of axioms:

1. $n + 5 = 8$.
2. $t - 3 = 7$.
3. $4V = 6 + 3V$.
4. $3x + 2 = x + 4$.
5. $7P - 1 = 2P - 11$.
6. $20k - 5 = 5k + 25$.
7. $7N + 13 = 12N - 35 + N$.
8. $\frac{a}{6} = 4$.
9. $\frac{x}{2} + \frac{x}{3} + x = 2$.
10. $\frac{2H}{5} - \frac{3H}{4} = 1 + \frac{H}{15}$.
11. $12 - \frac{4m}{5} + \frac{m}{12} = 2m$.

44. Equivalent Equations. — Two equations that have the same roots are called **equivalent equations**.

Thus, $2t - (5t + 5) = 7$, and $4t + 1 = t - 11$ are equivalent, because each is satisfied when $t = -4$, and for no other values of t.

It is evident that the process of solving an equation by use of the axioms in § 43 consists of deriving from the given equation an equation equivalent to it, then deriving from the new equation a third one equivalent to the second, etc., until finally an equation equivalent to the given equation is obtained that shows the value of the unknown number directly.

In general, if the same number expression is added to or subtracted from both members of an equation, an equivalent equation is derived. An equivalent equation is derived also if both members of an equation are multiplied or divided by any expression that is free of the unknown number. But, if both members are multiplied or divided by an expression containing the unknown number, the equation derived may not be equivalent to the given equation.

EXAMPLE 1.—Let $n + 3 = 5$. (1)

Multiplying by $n - 6$, $n^2 - 3n - 18 = 5n - 30$. (2)

Now, (1) is satisfied only when $n = 2$. But (2) is satisfied when $n = 2$ or when $n = 6$. Hence (2) is not equivalent to (1).

EXAMPLE 2.—Let $t^2 - 5t + 4 = 2t - 2$. (1)

Dividing by $t - 1$, $t - 4 = 2$. (2)

Now, (1) is satisfied when $t = 1$ or $t = 6$. But (2) is satisfied only when $t = 6$. Hence (2) is not equivalent to (1).

The above examples illustrate the following principles:

(1) *If both members of an equation are multiplied by an expression containing the unknown number, one or more extraneous roots may be introduced.*

(2) *If both members of an equation are divided by an expression containing the unknown number, one or more roots may be lost.*

Hence, in solving an equation, if both members are multiplied by an expression containing the unknown number, all roots obtained should be tested to see if they satisfy the original equation, and any root that does not should be discarded. And, to avoid losing roots, when the members of an equation are divided by an expression containing the unknown number, the roots obtained by equating the divisor to zero should be kept.

45. Transposition.—In the equation

$$a - b = c - d,$$

by adding b, $a = c - d + b$;

or adding d, $a - b + d = c$;

or subtracting a, $-b = c - d - a$;

or subtracting c, $a - b - c = -d$.

It is evident from the above equations that the effect of adding a term to, or subtracting a term from, each member of an equation is to make it disappear from its original position and to reappear in the other member of the equation with its sign changed.

The mechanical process of changing a term from either member of an equation to the other and changing its sign, which is equivalent to adding it to or subtracting it from both members, is called **transposition.** Hence,

A term of an equation may be transposed from either member to the other member by changing its sign.

46. Solution of Linear Equations in One Unknown Number. — To solve any linear equation in one unknown number:

(1) *If either member contains signs of grouping, remove them.*

(2) *Transpose all unknown terms to the left member and all known terms to the right member, changing the sign of each term transposed.*

(3) *Combine similar terms in each member.*

(4) *Divide each member by the resulting coefficient of the unknown number.*

(5) *Check the work by replacing the letter representing the unknown number in the original equation by the root found, and seeing if the two members have the same value.*

EXERCISES

Solve:

1. $5n - 4 = 3n + 8.$
2. $9V = 1 - V + 9.$
3. $4(x - 1) = 3(x + 1) - 4.$
4. $7(3 - 4A) = 5(4 - 3A).$
5. $5(R - 1) + 3(R + 1) = 0.$
6. $6t - (3t + 2) = 1.$
7. $N - (6 - 2N) = 9(N - 1).$
8. $2(12 + y) - (y - 3) = 0.$
9. $(P + 2)^2 - P^2 = P - 5.$
10. $5 + (a - 3)^2 = (a + 1)^2.$
11. $7(r + 2) - 6(r + 1) = 5r - 8(r + 3).$
12. $(x + 3)(x - 2) = (x - 5)(x + 1) + 29.$
13. $(K - 4)(K - 5) = (K + 9)(K - 8) + 2.$
14. $(b - 6)^2 = b(b - 22) + 136.$
15. $2y(y - 7) = (y + 8)(2y - 7).$
16. $(1 - 4B)(3B + 7) = 1 + (4B + 3)(5 - 3B).$

47. Fractional Equations. — To solve a fractional equation it is necessary first to clear it of fractions.

To clear an equation of fractions, multiply each member by the lowest common denominator of all fractions in the equation.

Caution. — Test all roots by substituting in the original equations to see that they were not introduced by clearing of fractions. (See (1), § 44, and (5), § 46.)

EXERCISES

Solve:

1. $\frac{1}{2a}+\frac{5}{8}=\frac{3}{a}$

2. $\frac{1}{x}+\frac{2}{x}=3-\frac{3}{x}.$

3. $\frac{m+4}{3m}=\frac{3m-8}{4m}.$

4. $\frac{4}{2t+6}=\frac{3}{4t-8}.$

5. $\frac{b}{b-1}-\frac{3}{b+1}=1.$

6. $\frac{V}{V+1}+2=\frac{3V}{V+2}.$

7. $\frac{2R+1}{2R-1}-\frac{2R-1}{1+2R}=\frac{8}{4R^2-1}.$

8. $\frac{t+1}{t-1}+\frac{t^2-3}{1-t^2}=\frac{4}{1+t}.$

9. $\frac{7}{a^2-3a-10}-\frac{2}{a^2+a-2}=\frac{12}{a^2-6a+5}.$

10. $\frac{k^2}{k^2+2k-15}=\frac{2k+10}{4k-12}+\frac{3k-1}{6k+30}.$

11. $\frac{1}{x-1}-\frac{1}{x-2}=\frac{1}{x-3}-\frac{1}{x-4}.$

Suggestion. — Perform the subtraction indicated in each member, before clearing of fractions.

12. $\frac{n-2}{n-8}-\frac{n-4}{n-10}=\frac{n-6}{n-12}-\frac{n-8}{n-14}.$

13. $\dfrac{W+7}{W+9}+\dfrac{W+15}{W+17}=\dfrac{W+13}{W+15}+\dfrac{W+9}{W+11}$.

14. $\dfrac{\dfrac{x}{x-1}-1}{\dfrac{x}{x-1}+1}=\dfrac{\dfrac{x}{x+1}-1}{\dfrac{x}{x+1}+1}$.

48. Literal Equations: Formulæ. — If an equation contains other letters than the one representing the unknown number, it is a **literal equation.** A literal equation may be solved by the same process as the equations in the preceding sections. The root, or value of the unknown number, will be an expression containing the other letters involved in the equation.

Many practical formulæ are solved as literal equations.

EXAMPLE. — The formula $C=\dfrac{nE}{R+nr}$ is used in electrical work. Solve it for n.

Clearing of fractions, $CR+nCr=nE$.
Transposing, $nCr-nE=-CR$.
Combining terms (factoring), $(Cr-E)n=-CR$.
Dividing by $Cr-E$, $n=\dfrac{-CR}{Cr-E}$ or $\dfrac{CR}{E-Cr}$

EXERCISES

1. Solve $5mx+n^2=m^2$ for x.

2. Solve $P(n+1)+P=n$ for n.

3. Solve $V^2+1+d(1-V)=V(V+d)$ for V.

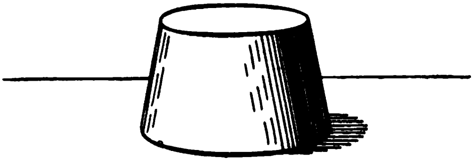
4. Solve $n=\dfrac{nx+y}{a}$ for x.

5. Solve $\dfrac{1}{A}+\dfrac{1}{B}=\dfrac{1}{C}+1$ for A.

6. Solve $\dfrac{at}{a^2-b^2}-\dfrac{3}{a-b}=\dfrac{t}{4a+4b}$ for t.

7. The area of a trapezoid is expressed by $A=\frac{1}{2}h(b+b')$. Solve for b.

8. The area of the conical surface of a frustum of a right circular cone is computed by the formula $S = \pi L(R + r)$. Solve for R; for r; for L.



9. The area of the entire surface of the solid in Ex. 8 is computed by the formula $S = \pi L(R + r) + \pi(R^2 + r^2)$. Solve for L.

10. The volume of the solid in Ex. 8 is computed by the formula $V = \frac{1}{3}\pi H(R^2 + r^2 + Rr)$. Solve for H.

11. A formula for the use of lenses is $\frac{1}{D} = \frac{1}{F - d} - \frac{1}{f}$. Solve for F; for f; for D; for d.

12. A formula used in electrical work is

$$\frac{1}{R} = \frac{1}{r_1} + \frac{1}{r_2} + \frac{1}{r_3} + \frac{1}{r_4} + \frac{1}{r_5}.$$

Solve for R; for r_1, etc.

13. In the formula $H = KND(D - 1)(R + 2)$, for computing the horse power of gasoline engines, solve for K; for N; for R.

14. In the formula $P = H\left(\frac{7.6}{460 + T} - \frac{7.9}{460 + t}\right)$, for computing the draft in a chimney, solve for H; for T; for t.

15. In the formula $\frac{VP}{273 + t} = \frac{V'P'}{273 + t'}$, which expresses the relation between the pressure, volume, and temperature of gases, solve for V; for V'; for P; for P'; for t; for t'.

49. **Problems.** — Some problems may be solved by use of linear equations in one unknown number. The procedure is as follows:

(1) *Let some letter represent one of the unknown numbers in the problem.*

(2) *Express in terms of this letter and the known numbers given or implied in the problem every quantity involved in the problem.*

(3) *Express by an equation the relation between these quantities that is either directly stated or implied in the problem.*

(4) *Solve the equation, and from the root found determine the values of all numbers sought.*

EXAMPLE. — Two grades of coffee costing the dealer 20 cents and 28 cents per pound, respectively, are to be mixed so that the blend shall be worth 25 cents per pound. How many pounds of each grade must be taken to make 50 pounds of the blend?

Let $n =$ number of pounds of 20-cent grade.
Then $50 - n =$ number of pounds of 28-cent grade.
$20n$ cents $=$ cost of n pounds of 20-cent grade.
$28(50 - n)$ cents $=$ cost of $50 - n$ pounds of 28-cent grade.
Hence $20n + 28(50 - n)$ cents $=$ cost of 50 pounds of blend.
But 50 pounds of blend cost 50×25 cents, or 1250 cents.
Hence, $20n + 28(50 - n) = 1250.$
Solving, $n = 18\frac{3}{4}.$
And $50 - n = 31\frac{1}{4}.$

EXERCISES

1. Find two consecutive even numbers whose product exceeds the square of the smaller by 48.

2. How many pounds each of two grades of coffee costing 20 cents and 28 cents, respectively, a pound, must a dealer use to make 100 pounds of a blend worth 25 cents a pound?

3. If one grade of coffee costs 24 cents a pound and another grade costs 30 cents a pound, how many pounds of each grade must be taken to make 9 pounds of a blend worth 28 cents a pound?

4. How many pounds each of tea worth 50 cents a pound and tea worth 80 cents a pound must be mixed to make 10 pounds of mixture worth 60 cents a pound?

5. A money drawer contains $2.40, consisting of nickels and dimes. There are 33 coins in all. How many coins of each kind are there?

6. A man made two investments together amounting to \$10,000. On the first investment he made annually 6 %, and on the other 5 %. His total income annually from both investments was \$535. Find the amount of each investment.

7. \$25,000 is invested in two places, part at 4 % and the rest at $5\frac{1}{2}$ %. The annual income from the two investments is \$1135. How much in each investment?

8. I invested \$12,000, part in 4 % bonds and the rest in a business enterprise that paid me 8 % annually. My annual income was \$800. Find the amount of each investment.

9. How many gallons of cream containing 26 % butter fat and milk containing 4 % butter fat must be mixed to make 60 gallons of milk testing 6 % butter fat?

10. How many gallons of cream containing 28 % butter fat and milk containing 4.5 % butter fat must be mixed to make 20 gallons of cream containing 25 % butter fat?

11. A certain solution contains 15 % alcohol. How many gallons of other ingredients must be added to 50 gallons of it so that the new mixture shall contain only 10 % alcohol?

12. Ordinary commercial alcohol is 95 % pure. How much water must be added to 20 quarts of it so that the diluted alcohol is only 80 % pure?

13. In an alloy of copper and tin weighing 24 pounds, 35 % is copper. How many pounds of copper must be added to this so that the new alloy is 40 % copper?

14. Tin appears to lose $\frac{2}{15}$ of its weight in air when immersed in water, and zinc appears to lose $\frac{1}{10}$ of its weight in air when immersed in water. If I find that a mass of alloy of these two metals weighing 50 pounds in air appears to weigh only 44 pounds in water, how many pounds of each metal in the alloy?

15. A man is 36 years old and his son 6 years old. In how many more years will the man be only twice as old as his son?

16. At what advance must a merchant mark an article costing him \$12 in order that he may then sell it at a reduction of 25 % from the marked price, and yet make a profit of 20 % on it?

17. A milliner advertises a lot of hats that cost her \$2.50 each for special sale at a reduction of 40 % from the regular marked price. At what advance must she first mark the hats in order that she may give this reduction and still make a profit of 20 %?

18. A teamster has one horse weighing 1200 pounds and another weighing 1500 pounds. If their draft power is proportional to their weight, how must he divide the 48-inch doubletree in order to justly distribute the load?

Suggestion. — The draft power of one multiplied by the number of units from the end of doubletree to the point of division equals the draft power of the other multiplied by the distance to the same point.

19. A messenger going 4 miles an hour has been gone 3 hours when it is found that the message is wrong. A second messenger, riding at an average of 6 miles an hour, is sent to overtake him. In how many hours will the second messenger overtake the first, and at what point?

20. An automobile was driven 125 miles in 4 hours 20 minutes. Part of the distance was in the country at a speed of 30 miles per hour, and the rest in the city at a speed of 15 miles per hour. Find how many hours of the drive were in the country.

21. A bullet with a velocity of 1650 feet per second is heard by the marksman to strike an object $2\frac{1}{2}$ seconds after it is fired. Sound travels approximately 1100 feet per second. How far away was the object that was hit?

22. Find the time between 2 and 3 o'clock when the hands of a clock are together.

Suggestion. — Represent by some letter the number of minute spaces on the clock face from the 12 o'clock point around to the point where the hands are together.

23. Find the time between 3 and 4 o'clock when the hands of a clock are opposite each other.

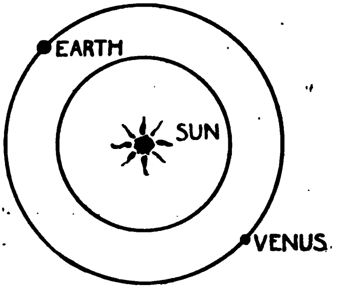
24. Find the time between 8 and 9 o'clock when the hands of a clock are opposite each other.

25. Find the time between 7 and 8 o'clock when the minute hand is 15 minute spaces ahead of the hour hand.

26. How long is it from the time that the hands of a clock are opposite each other until they are opposite each other again?

SUGGESTION. — Let t = number of minutes required. Show that $\frac{1}{60}t - \frac{1}{720}t = 1$.

27. If a planet and the earth are on opposite sides of the sun, and in a straight line with it, the planet is said to be *in opposition*. Venus makes a circuit around the sun in 225 days, and the earth makes a circuit in 365 days. If Venus is in opposition, how long will it be until it is in opposition again?



SUGGESTION. — Let n = the number of days required.

In n days the earth will make $\frac{n}{365}$ revolutions and Venus $\frac{n}{225}$ revolutions about the sun.

But to be again in opposition Venus must make one more revolution than the earth.

Hence, $$\frac{n}{225} - \frac{n}{365} = 1.$$

28. Mars makes a circuit about the sun in 687 days. Mars was in opposition November 11, 1911. Find the date upon which it was again in opposition.

29. Mercury makes a circuit about the sun in 88 days. How long is it from the time that it is in opposition until it is in opposition again?

30. Jupiter makes a circuit about the sun in 4333 days. How long is it from the time that it is in opposition until it is in opposition again? It was in opposition February 28, 1909. Find the next date of opposition.

31. In a factory a large machine can turn out a number of articles in 6 days, and a smaller sized machine can turn them out in 9 days. If both machines are operated at once, how long will it take them to turn out the articles?

SUGGESTION. — If $n =$ the number of days required, show that

$$\frac{1}{n}=\frac{1}{6}+\frac{1}{9}.$$

32. There are two large machines and three small ones in a factory. To manufacture a given order of goods would take either one of the large machines alone 20 days, and either one of the small machines alone 35 days. If all five machines are operated, how long does it take to fill the order?

33. A water tank can be filled by one pipe in 5 hours and emptied by another in 8 hours. If both pipes are open, how long will it take to fill the tank?

34. In a city waterworks plant are two steam pumps. One pump would fill the reservoir in 15 hours, and the other would fill it in 18 hours. If both pumps are operated at once, how long does it require to fill the reservoir?

35. An aviator flew to a point 75 miles away, and back in 5 hours, 30 minutes. His rate returning was 30 miles per hour. What was his rate going?

36. An aviator made 30 miles per hour against the wind on a trip of 40 miles. After he had gone 25 miles, the wind ceased. The entire time required for the trip was 1 hour 10 minutes. What was the velocity of the wind?

37. A boatman rows 5 miles in $\frac{3}{4}$ hour with the current and it takes him $1\frac{3}{4}$ hours to return the same distance against the current. Find the rate of the current.

38. Find a number such that if its pth part be added to it, the sum will be equal to the qth part of it plus r.

SUPPLEMENTARY EXERCISES

Solve:

1. $\frac{2n+1}{2n-1}-\frac{2n-1}{1+2n}=\frac{8}{4n^2-1}$.

2. $\frac{2A+1}{2A-16}-\frac{2A-1}{12+2A}-\frac{17+9A}{A^2-2A-48}=0$.

3. $\frac{x-1}{x-2}-\frac{x-2}{x-3}=\frac{x-3}{x-4}-\frac{x-4}{x-5}$.

4. $\frac{x-4}{x-5}-\frac{x-5}{x-6}=\frac{x-7}{x-8}-\frac{x-8}{x-9}$.

5. $\frac{t-8}{t-6}+\frac{t+1}{t-1}-\frac{t}{t-2}=\frac{t-9}{t-7}$.

6. Solve $\frac{ax^2}{b-cx}+a+\frac{ax}{c}=0$ for x.

7. Solve $\frac{3}{an-ab}-\frac{b}{b^2-bn}-\frac{9+3a}{2ab}=0$ for n.

8. The formula $R=\dfrac{\frac{3pV}{2T}+\frac{pV}{T}+E}{\frac{3pV}{2T}+E}$ expresses a law of science relating to matter, energy, and heat. Solve it for V; for p; for E; for T.

9. A train leaves a station and runs at a speed of 35 miles an hour. Two hours later a second train leaves the same station and follows the first train at a speed of 45 miles an hour. In how many hours will the second train pass the first?

10. A number is composed of two digits whose sum is 10. This number exceeds by 54 the number obtained by interchanging the digits. Find the number.

11. An express train traveling 42 miles per hour started from the same depot 50 minutes after a freight train, which it overtook in 2 hours, 5 minutes. Find the velocity of the freight train.

12. A messenger was sent to carry a despatch. Five hours after his departure a second messenger was sent, who ought to overtake the first one in 8 hours. To do this he was obliged to travel $2\frac{1}{2}$ miles per hour faster. How many miles per hour did the first travel?

13. Two trains move in the same direction. The first starts from A at 2 o'clock in the morning, and travels 32 miles per hour. The second starts from B at 8:30 o'clock in the morning, and travels 42 miles per hour. The distance between A and B is 80 miles. When and where does the second overtake the first?

14. A decimal, such as 0.343434 ⋯, in which certain figures repeat themselves indefinitely, is called a **repeating decimal.** Every repeating decimal is equivalent to some common fraction. Thus 0.6666 ⋯ is equal to $\frac{2}{3}$.

Find the common fraction to which 0.272727 ⋯ is equivalent.

Solution. — The figures 27 repeat. Hence we separate the number into two parts, thus: 0.272727 ⋯ = 0.27 + 0.002727 ⋯. It is clear that the second term is just $\frac{1}{100}$ of the whole decimal. Hence, if v represents the value of the fraction sought,

$$v = 0.27 + \frac{v}{100}$$

Solving

$$v = \frac{3}{11}.$$

15. Find the common fraction to which 0.5135135 ⋯ is equivalent.

Suggestion. — $v = 0.513 + \frac{v}{1000}$.

Find the common fraction to which each of the following is equivalent:

16. 0.0727272 ⋯.
17. 0.121212 ⋯.
18. 0.060606 ⋯.
19. 0.123123 ⋯.
20. 0.818181 ⋯.
21. 0.432432 ⋯.
22. 0.522522 ⋯.
23. 0.61716171 ⋯.
24. 1.66666 ⋯.

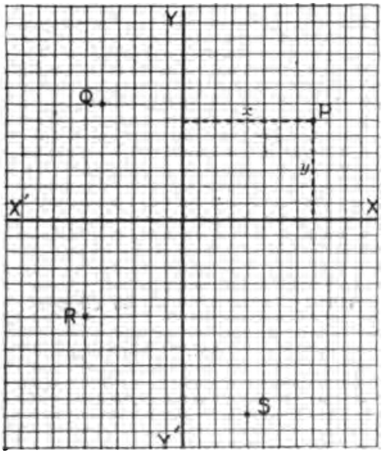
CHAPTER V.

SYSTEMS OF LINEAR EQUATIONS

50. Equations in Two Unknown Numbers. — A linear equation containing two unknown numbers is satisfied by an unlimited number of *sets of values* of the unknown numbers. For any particular value that may be assigned to either one of the unknown numbers, one particular value of the other can be found for which the equation is true.

Thus, in $2x - 3y = 24$, by assigning values to y and solving for the corresponding values of x, we find that when $y = 0$, $x = 12$; when $y = 1$, $x = 13\frac{1}{2}$; when $y = 2$, $x = 15$; when $y = 4$, $x = 18$; when $y = 6$, $x = 21$; when $y = -1$, $x = 10\frac{1}{2}$; when $y = -2$, $x = 9$; when $y = -4$, $x = 6$; etc. The number of sets of values may be continued indefinitely.

51. Graph of a Linear Equation in Two Unknown Numbers. — The relation between two unknown numbers which is expressed by an equation may be shown also by means of a **graph.** The construction of the graph of a linear equation in two unknown numbers was discussed in the FIRST COURSE, § 124.



Just as a point on the earth's surface is located by knowing its directions and distances from the prime meridian and the equator (longitude and latitude), so if two lines XX' and YY' are drawn at right angles on squared paper, the position of any point P on the paper is determined by its directions and distances from these lines. The line XX' is called the **x-axis,** YY' the **y-axis,** and their point of intersection the **origin.** The distance x (measured parallel to XX') of P from the y-axis and the distance y (measured parallel to YY') of P from the x-axis are called the **coördinates** of P. Distances measured to the right of YY' are *positive,* and to the left *negative;* and distances measured upward from XX' are *positive,* and downward *negative.*

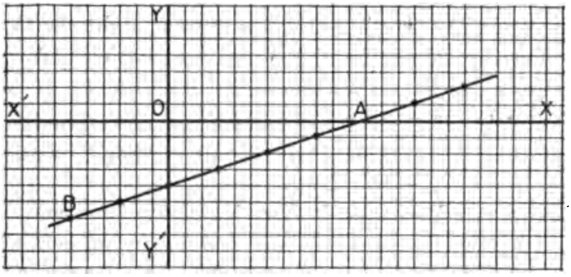
Thus, the coördinates of P are $x = +8$ and $y = +6$; of Q, $x = -5$ and $y = +7$; of R, $x = -6$ and $y = -6$; and of S, $x = +4$ and $y = -12$.

To draw the graph of a linear equation in x and y, a number of sets of values of x and y that satisfy the equation are found as in § 50. Axes are drawn, and points are located, the coördinates of each of which correspond to one of the sets of values found for x and y. A smooth line is drawn connecting these points. This line is the **graph of the equation.**

EXAMPLE. — Draw the graph of $x - 3y = 12$.

Assigning values to y and solving for x, the following sets of values of x and y are found:

x	12	15	18	9	6	3	0	-3	-6
y	0	1	2	-1	-2	-3	-4	-5	-6



Locating the points whose coördinates correspond to these sets of values of x and y, and connecting them, gives the straight line AB shown in the figure.

It may be shown that,

The graph of every linear equation in two unknown numbers is a straight line.

Hence, to draw the graph of such an equation it is sufficient to locate only *two points* of it, and draw the straight line through them with a ruler. The two points should be taken a good distance apart for accuracy in drawing.

EXERCISES

1. By assigning values to y and computing the corresponding values of x, find ten sets of values of x and y that satisfy $3x + 2y = 24$. Locate the ten points whose coördinates correspond to these values. Join the points.

By locating two points of each, and drawing the straight line through them with a ruler, construct the graphs of the following:

2. $x - 2y = 8.$
3. $4x + y = 12.$
4. $5x - y = 20.$
5. $2x + 5y = 15.$
6. $8x - 3y = 32.$
7. $x + 6y = 18.$
8. $10x + y = 30.$
9. $4x - 7y = 20.$
10. $3x + 24 = y.$

11. If C is the temperature as indicated on a Centigrade thermometer, and F the temperature as indicated on a Fahrenheit thermometer, then $C = \frac{5}{9}(F - 32)$. Draw the graph of this equation.

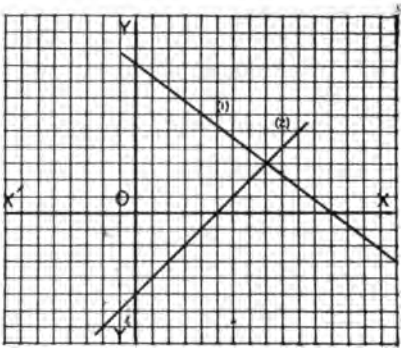
12. By using the graph in Ex. 11, give at sight the Centigrade readings corresponding to the following Fahrenheit readings: 40° F.; 12° F.; 18° F.; − 10° F.; − 20° F.; − 5° F.

Give at sight the Fahrenheit readings corresponding to the Centigrade readings: 34° C.; 15° C.; 40° C.; 8° C.; − 10° C.

52. Systems of Equations in Two Unknown Numbers. — Two or more equations containing the same unknown numbers are called a **system of equations.** As shown in the FIRST COURSE (§ 131), in a system of two linear equations in two unknown numbers there is, in general, one and only one set of values of the unknown numbers that satisfy both equations. Such a set of values is called the **solution of the system.**

Thus, in the system $\begin{cases} x + 2y = 8, \\ 3x - y = 10, \end{cases}$ the values $x = 4$ and $y = 2$ satisfy both equations. Hence, the solution of the system is $x = 4$, $y = 2$.

53. Systems solved Graphically. — The solution of a system of two linear equations in two unknown numbers may be found by drawing the graphs of the equations, and observing the coördinates of the point where the graphs meet. Since this point is on both graphs, its coördinates must satisfy both equations.



Example. — Solve

$$\begin{cases} 3x + 4y = 36, & (1) \\ x - y = 5. & (2) \end{cases}$$

The graphs of equations (1) and (2) are lines (1) and (2), respectively. They meet at a point whose coördinates are $x = 8$, $y = 3$. Hence, the solution is $x = 8$, $y = 3$.

EXERCISES

Solve graphically:

1. $\begin{cases} x - y = 4, \\ x + y = 8. \end{cases}$

2. $\begin{cases} x + 4y = 10, \\ 3x - y = 12. \end{cases}$

3. $\begin{cases} x + y = 5, \\ 2x + 3y = 12. \end{cases}$

4. $\begin{cases} 5x + 4y = 22, \\ 3x + y = 9. \end{cases}$

5. $\begin{cases} 7x - 2y = 31, \\ 4x + 3y + 3 = 0. \end{cases}$

6. $\begin{cases} 3x + 2y = 12, \\ 4x - 3y + 1 = 0. \end{cases}$

7. $\begin{cases} 2x + 5y = 28, \\ 5x - 8y = 29. \end{cases}$

8. $\begin{cases} 7x + y = 42, \\ 3x - y = 8. \end{cases}$

9. $\begin{cases} x + y = 30, \\ 3x - 2y = 25. \end{cases}$

10. By drawing the graphs of the equations, show that

$$\begin{cases} x - 2y = 8, \\ 2x - 4y = 40, \end{cases}$$

has no solution. What kind of lines are the graphs?

54. Systems solved by Elimination. — A system of equations may be solved by a process called **elimination.** This process, applied to a system of two linear equations in two unknown numbers, consists of combining the members of the two equations so as to derive a new equation which contains only one unknown number. The new equation is then solved. By substituting the value of the unknown number found in either of the original equations, and solving, the value of the other unknown number is obtained.

The three most common methods of elimination, as shown in the FIRST COURSE, are as follows:

ELIMINATION BY ADDITION OR SUBTRACTION

To eliminate one of the unknown numbers, multiply the members of each equation, if necessary, by such a number as will make the numerical coefficients of that unknown number the same in both of the resulting equations. Add or subtract (according to signs) the corresponding members of the resulting equations.

EXAMPLE. — Solve $\begin{cases} 2M + 5N = 15, & (1) \\ 3M - 4N = 11. & (2) \end{cases}$

Let us eliminate N.

Multiplying (1) by 4, $8M + 20N = 60.$ (3)

Multiplying (2) by 5, $15M - 20N = 55.$ (4)

Adding (3) and (4), $23M = 115.$

Solving, $M = 5.$

Substituting 5 for M in (1), $10 + 5N = 15.$

Solving, $N = 1.$

ELIMINATION BY COMPARISON

Solve each of the equations for the value of either one of the unknown numbers, expressed in terms of the other, and write one of the values equal to the other. Solve the resulting equation.

EXAMPLE. — Solve $\begin{cases} 6v - 5t = 1, & (1) \\ 9v + 10t = 12, & (2) \end{cases}$

Let us eliminate v.

Solving (1) for v, $v = \frac{1 + 5t}{6}.$ (3)

Solving (2) for v, $$v = \frac{12 - 10t}{9}. \qquad (4)$$

From (3) and (4), $$\frac{1 + 5t}{6} = \frac{12 - 10t}{9}.$$

Solving, $t = \frac{3}{5}$.

Substituting $\frac{3}{5}$ for t in (1), $6v - 3 = 1$.

Solving, $v = \frac{2}{3}$.

ELIMINATION BY SUBSTITUTION

Solve one of the equations for the value of either unknown number, expressed in terms of the other, and substitute this value in place of that number in the other equation. Solve the resulting equation.

EXAMPLE.—Solve $$\begin{cases} 4a - 5b = 26, & (1) \\ 3a - 6b = 15. & (2) \end{cases}$$

Let us eliminate a.

Solving (1) for a, $$a = \frac{26 + 5b}{4}. \qquad (3)$$

Substituting the value of a in (3) for a in (2),

$$\frac{3(26 + 5b)}{4} - 6b = 15. \qquad (4)$$

Solving (4), $b = 2$.

Substituting 2 for b in (1), $4a - 10 = 26$.

Solving, $a = 9$.

EXERCISES

Eliminate by addition or subtraction, and solve:

1. $\begin{cases} 7m + n = 42, \\ 3m - n = 8. \end{cases}$

2. $\begin{cases} 2A - B = 5, \\ 5A - 2B = 14. \end{cases}$

3. $\begin{cases} 2x + 5y = 15, \\ 3x - 4y = 11. \end{cases}$

4. $\begin{cases} 5v - 2t = 35, \\ t + 4v = 25. \end{cases}$

5. $\begin{cases} 4R - 3r = 1, \\ 3R - 4r = 6. \end{cases}$

6. $\begin{cases} 8V = t + 34, \\ V + 8t = 53. \end{cases}$

Eliminate by comparison and solve:

7. $\begin{cases} 2p - q = 9, \\ 5p - 3q = 14. \end{cases}$

8. $\begin{cases} 3M + 2N = 2, \\ M - N = 9. \end{cases}$

9. $\begin{cases} 3x - y = 1, \\ 2x + 5y = 41. \end{cases}$

10. $\begin{cases} 9a - 5b = 13, \\ 5a + b = 11 \end{cases}$

11. $\begin{cases} 2E - F = 5, \\ E + 2F = 25. \end{cases}$

12. $\begin{cases} 10m + 3n = 174, \\ 3m + 10n = 125. \end{cases}$

Eliminate by substitution and solve:

13. $\begin{cases} a+b=30, \\ 3a-2b=25. \end{cases}$

14. $\begin{cases} 4w=5v+26, \\ 3w-6v=15. \end{cases}$

15. $\begin{cases} 3r+7r'=16, \\ 2r+5r'=13. \end{cases}$

16. $\begin{cases} 7F+4C=1, \\ 9F+4C=3. \end{cases}$

17. $\begin{cases} 8A+2B=90, \\ 3A-5B=51. \end{cases}$

18. $\begin{cases} 5y+11z=102, \\ y-3z+16=0. \end{cases}$

Eliminate by any method and solve:

19. $\begin{cases} 7D-9T=13, \\ 5D+2T=10. \end{cases}$

20. $\begin{cases} 18k-20s=1, \\ 15k+16s=9. \end{cases}$

21. $\begin{cases} 8x+6y=10, \\ 5x+2y=1. \end{cases}$

22. $\begin{cases} 7a-4b=12, \\ 8a=5b. \end{cases}$

23. $\begin{cases} \frac{1}{3}a-\frac{1}{4}b=2, \\ \frac{1}{4}a-\frac{1}{2}b=7. \end{cases}$

24. $\begin{cases} \frac{1}{2}m+\frac{1}{3}n=13, \\ \frac{1}{5}m+\frac{1}{8}n=5. \end{cases}$

Solve for x and y:

25. $\begin{cases} ax-by=a^2-b^2, \\ bx-ay=a^2-b^2. \end{cases}$

26. $\begin{cases} ax+by=a^2+b^2, \\ bx+ay=2ab. \end{cases}$

27. $\begin{cases} x-ay=a+b, \\ ax-ay=a^2+b. \end{cases}$

28. $\begin{cases} x+ny=m, \\ 2x-my=n. \end{cases}$

29. $\begin{cases} x+p^2y=0, \\ x-q^2y=1. \end{cases}$

30. $\begin{cases} ax+by=c, \\ dx+ey=f. \end{cases}$

Eliminate before clearing of fractions, and solve:

31. $\begin{cases} \frac{2}{x}-\frac{1}{y}=9, \\ \frac{5}{x}-\frac{3}{y}=14. \end{cases}$

32. $\begin{cases} \frac{4}{a}-\frac{3}{b}=1, \\ \frac{3}{a}-\frac{4}{b}=6. \end{cases}$

33. $\begin{cases} \frac{5}{A}-\frac{2}{B}=1, \\ \frac{8}{A}+11=\frac{5}{B}. \end{cases}$

34. $\begin{cases} \frac{5}{m}+\frac{2}{n}=1, \\ \frac{13}{m}+\frac{8}{n}=11. \end{cases}$

35. $\begin{cases} \frac{3}{v}+\frac{2}{t}=2, \\ \frac{1}{v}-\frac{1}{t}=9. \end{cases}$

36. $\begin{cases} \frac{4}{M}+\frac{1}{N}=34, \\ \frac{4}{N}+\frac{1}{M}=16. \end{cases}$

55. Systems in Three or More Unknown Numbers. — The process of elimination described in § 54 may be applied in the solution of a system of three equations in three unknown numbers, a system of four equations in four unknown numbers, etc. In any case, the equations of the system are combined so as to derive a new system in which the number of equations is one less than the number in the given system, and in which one of the unknown numbers does not appear. By repeating the process, a new system is obtained in which the number of equations and the number of unknown numbers is one less than before, etc. Finally a single equation is obtained which contains only one unknown number.

EXAMPLE. — Solve
$$\begin{cases} 2a + b - c + 2d = 11, & (1) \\ 2a - b + 2c + d = 2, & (2) \\ a + b + c + 3d = 5, & (3) \\ 5a - 2b + c - d = 10. & (4) \end{cases}$$

Eliminating d between (1) and (2) gives

$$2a - 3b + 5c = -7. \qquad (5)$$

Eliminating d between (1) and (3) gives

$$4a + b - 5c = 23. \qquad (6)$$

Eliminating d between (1) and (4) gives

$$12a - 3b + c = 31. \qquad (7)$$

Now (5), (6), and (7) form a new system free of d.

Eliminating c between (5) and (6) gives

$$6a - 2b = 16. \qquad (8)$$

Eliminating c between (5) and (7) gives

$$58a - 12b = 162. \qquad (9)$$

Now (8) and (9) form a new system free of c and d.

Eliminating b between (8) and (9) gives

$$22a = 66. \qquad (10)$$

Solving (10), $a = 3.$

Substituting 3 for a in (8), $b = 1.$

Substituting 3 for a and 1 for b in (5), $c = -2.$

Substituting 3 for a, 1 for b, and -2 for c in (1), $d = 1.$

EXERCISES

Solve:

1. $\begin{cases} a+b+c=6, \\ 3a-b+2c=7, \\ 2a+3b-c=5. \end{cases}$

2. $\begin{cases} x+y+z=9, \\ 27x+9y+3z=93, \\ 8x+4y+2z=36. \end{cases}$

3. $\begin{cases} 2M-3N-7P=19, \\ 5M-8N+11P=24, \\ 7M-11N+4P=43. \end{cases}$

4. $\begin{cases} r+2s+3t=14, \\ 4r-5s+6t=12, \\ r+15s-9t=58. \end{cases}$

5. $\begin{cases} 3A+2C=6-2B, \\ A-5B+6C=2, \\ 6A-8C=12. \end{cases}$

6. $\begin{cases} x+y=1, \\ y+z=9, \\ x+z+6=0. \end{cases}$

7. $\begin{cases} 2a+b-c=7, \\ b-a=1, \\ c-b=1. \end{cases}$

8. $\begin{cases} x+y+z+w=10, \\ x-y-z+w=0, \\ 2x+y+3z-w=9, \\ 3x-y+z-2w+4=0. \end{cases}$

9. $\begin{cases} 2p-q-r+s=13, \\ p+q-r-s=-1, \\ p-q+r-s+5=0, \\ 3p+2q-r+2s=17. \end{cases}$

10. $\begin{cases} 4E-G+K=0, \\ 7G+2K+H=0, \\ 4K+H+8E=0, \\ 5G+16E-H=4. \end{cases}$

11. $\begin{cases} \frac{2}{a}+\frac{5}{b}-\frac{7}{c}=9, \\ \frac{5}{a}-\frac{1}{b}+\frac{3}{c}=16, \\ \frac{7}{a}+\frac{6}{b}+\frac{1}{c}=34. \end{cases}$

12. $\begin{cases} \frac{1}{x}+\frac{2}{y}=26, \\ \frac{3}{x}+\frac{4}{z}=56, \\ \frac{5}{y}+\frac{6}{z}=65. \end{cases}$

Solve for x, y, and z:

13. $\begin{cases} x+y+z=4a, \\ x-y+2z=4a+2b, \\ 2x+y-z=a+b. \end{cases}$

14. $\begin{cases} ax+by+cz=p, \\ dx+ey+fz=q, \\ gx+hy+iz=r. \end{cases}$

56. Number of Equations and Unknown Numbers in a System. — If a system of linear equations contains just as many equations as there are unknown numbers, in general *one* and *only one solu-*

tion of the system can be found. Such a system is called a **determinate system.** The systems solved in § 55 were *determinate systems.*

If there are more unknown numbers than equations in a system, in general the number of solutions of the system is *indefinitely great.* Hence, such a system is called an **indeterminate system.**

Thus, $\begin{cases} x + y + z = 6, \\ 2x - y + z = 3, \end{cases}$ is *indeterminate.* Some of its many solutions are $x = 1, y = 2, z = 3$; $x = 3, y = 3, z = 0$; $x = -1, y = 1, z = 6$; etc.

If there are more equations than unknown numbers in a system, in general no set of values of the unknown numbers exists which will satisfy all equations of the system. Hence, such a system is called an **impossible system.**

Thus, in $\begin{cases} m - n = 5, \\ m + 2n = 14, \\ 2m - 3n = 4, \end{cases}$ the only values of m and n that satisfy the first two equations are $m = 8$ and $n = 3$. But these values do not satisfy the third equation.

There are exceptions to some of the above principles in cases in which special relations exist between the equations of a system.

Thus, the system $\begin{cases} 2x - y = 3, \\ 4x = 6 + 2y \end{cases}$ is *not determinate.* Any values of x and y that satisfy either of the equations satisfy the other also. The equations are not independent. The second may be derived from the first by multiplying by 2 and transposing a term.

Two equations in which all values of the unknown numbers that satisfy one satisfy the other also are called **equivalent equations.**

The system $\begin{cases} 2x + 3y = 14, \\ 3x - 2y = 8, \\ 5x + y = 22, \end{cases}$ is *not impossible,* for each equation is satisfied when $x = 4$ and $y = 2$. The equations are not independent, for the third is derived from the other two by addition.

The system $\begin{cases} x + 5y = 3, \\ 2x + 10y = 15, \end{cases}$ has *no solution.* If the first is multiplied by 2 and subtracted from the second, it gives $0 = 9$, which is absurd. Hence, the system is *not determinate.*

Two equations which are not satisfied by any set of values of the unknown numbers are called **inconsistent equations.**

EXERCISES

1. Show that the equations $x+y=7$, $2x-y=8$, $x-2y=1$ do not form an impossible system. What is the solution?

2. Draw the graphs of the three equations in Ex. 1 upon the same set of axes. What do you observe about the points where they meet?

3. Show that $3m+n-2=0$ and $3n=6-9m$ are equivalent. How may the second be obtained from the first?

4. Draw the graphs of the two equations in Ex. 3. What do you observe about them?

5. Show that the system $x-2y=7$ and $3x=36+6y$ has no solution.

6. Draw the graphs of the equations in Ex. 5. What do you observe about them?

57. Problems solved by Systems of Linear Equations. — If a problem contains more than one unknown number, each of these may be represented by some letter, and from relations given in the problem, as many equations containing these letters may be formed as there are unknown numbers. By solving the system of equations thus obtained, the values of all unknown numbers in the problem are found directly.

EXERCISES

1. Of two partners, A and B, A has twice as much invested in a business as B. If the profits from the business are $3600, how much should each receive?

2. A man invested part of $10,000 at 5 % and the rest at 8 %. His annual income from both investments was $608. Find the amount of each investment.

3. A man invested $24,000, part in municipal 4 % bonds, and the rest in railroad stock that yielded him 7 %. His annual

income from the entire investment was $1230. Find the amount invested in bonds and the amount invested in stocks.

4. A dealer wishes to mix coffee costing him 20 cents a pound with coffee costing 30 cents a pound so as to make a blend that will cost him 24 cents a pound. How many pounds of each grade must he use to make 50 pounds of the blend?

5. A tea merchant has one grade of tea costing 40 cents a pound and another grade costing 65 cents a pound. How many pounds of each grade must he use to make 20 pounds of a mixture costing him 50 cents a pound?

6. How much cream containing 22 % butter fat and milk containing 4 % butter fat must be mixed to make 40 gallons of cream containing 20 % butter fat?

7. How much milk testing $3\frac{1}{2}$ % butter fat and cream testing 24 % butter fat must be mixed to make 100 gallons of milk testing 4 % butter fat?

8. A certain mixture contains 30 % alcohol. Commercial alcohol is 95 % pure. How many gallons each of the mixture and commercial alcohol must be taken to make 60 quarts of a mixture which contains 45 % alcohol?

9. A chemist has in his laboratory the same acid in two strengths. Six parts of one mixed with 4 parts of the other gives a mixture 86 % pure, and 4 parts of the first mixed with 6 parts of the second gives a mixture 84 % pure. What is the per cent of purity of each acid?

10. When weighed in water, gold appears to lose $\frac{4}{77}$ of its weight in air. Silver appears to lose $\frac{2}{21}$ of its weight in air when weighed in water. An alloy of gold and silver that weighs 18 ounces in air appears to weigh only $16\frac{16}{21}$ ounces in water. How much gold and how much silver in the alloy?

11. Hieron, king of Syracuse, had a crown that weighed 20 pounds. The workman who made it claimed that it was of pure gold, but King Hieron suspected that it was an alloy of gold and silver. His friend Archimedes, the greatest mathematician of

antiquity, weighed it in water and found that it appeared to weigh only $18\frac{3}{4}$ pounds. How much gold and how much silver did it contain? (Use the data of Problem 10.)

12. When weighed in water tin appears to lose $\frac{2}{15}$ of its weight in air, and zinc $\frac{1}{10}$ of its weight in air. An alloy of tin and zinc that weighed 45 pounds in air appeared to weigh only $39\frac{1}{2}$ pounds in water. Find the amount of each metal in the alloy.

13. The relation between the readings on the Centigrade and the Fahrenheit thermometers is expressed by $C=\frac{5}{9}(F-32)$, where C and F are the temperatures indicated on the Centigrade and Fahrenheit thermometers, respectively. Alcohol boils at a temperature that is 96° lower on the Centigrade than on the Fahrenheit thermometer scale. Find the boiling points of alcohol on both the Centigrade and Fahrenheit thermometers.

14. The temperature of boiling mercury is 312° higher when read upon the Fahrenheit thermometer than when read upon the Centigrade. Find the boiling point of mercury on each scale.

15. Two men lift a weight of 216 pounds which is suspended from a 6-foot pole carried between them. If the weight is $2\frac{1}{2}$ feet from one man and $3\frac{1}{2}$ feet from the other, how much does each man lift?

NOTE. — The equations for Problems 15 and 16 depend upon the principle that the amount lifted by one multiplied by the distance to the weight equals the amount lifted by the other multiplied by the distance to the weight.

16. Two boys carry a basket of groceries suspended on a stick between them. If the basket weighs 60 pounds, and is placed 24 inches from one end and 30 inches from the other end of the stick, how much does each boy carry if one grasps each end of the stick?

17. An aviator flew to a point 30 miles away in 40 minutes, with the wind. It took 2 hours for him to make the return trip against the wind. Find the speed of the wind and what his speed would have been if there had been no wind.

18. A freight train is 715 feet long, and a passenger train 275 feet long. They pass each other on parallel tracks. When going in opposite directions, it takes them just 9 seconds to pass; but when going in the same direction, it takes them 45 seconds to pass. Find the speed of each train.

19. A man said to his son: "I am now 4 times as old as you; but in 20 years I will be only twice as old as you." Find the age of each.

20. A pumping station contains two steam pumps. If both pumps are driven at once, they will fill the reservoir in $8\frac{8}{9}$ hours. If one is driven 10 hours and the other 8 hours, the reservoir will be filled. In what time can each pump alone fill the reservoir?

21. Two machines operated together can turn out a piece of work in 13 hours 20 minutes. After both have been operated for 6 hours, one machine is stopped, and the other finishes the work in $16\frac{1}{2}$ hours more. In how many hours could each machine alone turn out the work?

22. The perimeter (sum of the sides) of a triangle is 36 inches. One side is 3 inches longer than another, and the third side is 3 inches longer than the longer of the other two. Find the length of each side.

23. The sum of the two sides of a triangle which meet at one vertex is 28 inches, at another vertex 34 inches, and at the third vertex 38 inches. Find the length of each side.

24. Three circles are drawn tangent to (touching) each other, and with their centers at the points P, Q, and R, respectively. The distance between P and Q is 12 inches, between P and R 16 inches, and between Q and R 20 inches. Find the radii of the circles.

NOTE. — The lines PQ, QR, and PR pass through the points at which the circles touch.

25. The sides of a triangle are all tangent to a circle. The sides of the triangle are 28 inches, 36 inches, and 40 inches respectively. Find the points of the sides at which they touch the circle.

NOTE. — Use the fact from geometry that the distances from any vertex of the triangle to the two points where the sides from that vertex touch the circle are equal.

26. Three pumps together can fill a reservoir in $3\frac{3}{23}$ hours, the two larger can fill it in $4\frac{4}{17}$ hours, and the largest and smallest together can fill it in $4\frac{4}{5}$ hours. In what time could each pump alone fill it?

58.* Determinants. — If we solve the system

$$\begin{cases} ax + by = c, \\ dx + ey = f, \end{cases}$$

the solution is
$$x = \frac{ce - bf}{ae - bd},\quad y = \frac{af - cd}{ae - bd}.$$

It is seen that the fractions expressing the values of x and y have the same denominator, $ae - bd$. This denominator may be written in the form

$$\begin{vmatrix} a & b \\ d & e \end{vmatrix}.$$

This symbol is called a **determinant**.

Its value is found by subtracting from the product of the upper left-hand and lower right-hand numbers the product of the upper right-hand and lower left-hand numbers.

Thus, $\begin{vmatrix} 6 & 2 \\ 5 & 7 \end{vmatrix} = 6 \times 7 - 5 \times 2 = 32.$ And $\begin{vmatrix} -8 & 3 \\ 4 & 1 \end{vmatrix} = -8 - 12 = -20.$

Since the above determinant contains *two rows* and *two columns* of numbers, it is called a determinant of the **second order**.

EXERCISES

Find the value of:

1. $\begin{vmatrix} 5 & 7 \\ 2 & 8 \end{vmatrix}$.
2. $\begin{vmatrix} 1 & 6 \\ 6 & 1 \end{vmatrix}$.
3. $\begin{vmatrix} 4 & 3 \\ 3 & 2 \end{vmatrix}$.
4. $\begin{vmatrix} 10 & 3 \\ 8 & 6 \end{vmatrix}$.

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* This and the following starred sections may be omitted without interfering with the subsequent work, if desired.

5. $\begin{vmatrix} -2 & 3 \\ 4 & -1 \end{vmatrix}$. 7. $\begin{vmatrix} 8 & 0 \\ 3 & 2 \end{vmatrix}$. 9. $\begin{vmatrix} 12 & -2 \\ -9 & -1 \end{vmatrix}$. 11. $\begin{vmatrix} x & x \\ 1 & 3 \end{vmatrix}$.

6. $\begin{vmatrix} 12 & -2 \\ 1 & -1 \end{vmatrix}$. 8. $\begin{vmatrix} 0 & -2 \\ 5 & 4 \end{vmatrix}$. 10. $\begin{vmatrix} a & b \\ c & d \end{vmatrix}$. 12. $\begin{vmatrix} a^2 & 1 \\ 1 & 1 \end{vmatrix}$.

59.* Systems of Two Linear Equations solved by Determinants.[†] —It follows from the definition of a determinant in § 58 that the solution of the system $\begin{cases} ax + by = c \\ dx + ey = f \end{cases}$ may be written in the form

$$x = \frac{\begin{vmatrix} c & b \\ f & e \end{vmatrix}}{\begin{vmatrix} a & b \\ d & e \end{vmatrix}}, \qquad y = \frac{\begin{vmatrix} a & c \\ d & f \end{vmatrix}}{\begin{vmatrix} a & b \\ d & e \end{vmatrix}}.$$

Observe that the numerator of x *is obtained from the denominator by replacing the coefficients of* x *that form the first column by the known terms in the equations, and that the numerator of* y *is obtained by replacing the coefficients of* y *that form the second column by the known terms in the equations.*

Hence, instead of solving by the process of elimination, the solution of any system of two linear equations in two unknown numbers may be *written down* by determinants.

EXAMPLE. — Solve $\begin{cases} 4A - 3B = 1, \\ 3A - 4B = 6. \end{cases}$

$$A = \frac{\begin{vmatrix} 1 & -3 \\ 6 & -4 \end{vmatrix}}{\begin{vmatrix} 4 & -3 \\ 3 & -4 \end{vmatrix}} = \frac{14}{-7} = -2, \qquad B = \frac{\begin{vmatrix} 4 & 1 \\ 3 & 6 \end{vmatrix}}{\begin{vmatrix} 4 & -3 \\ 3 & -4 \end{vmatrix}} = \frac{21}{-7} = -3.$$

[†] The determinant notation is an invention of modern times. It has been in use in solving systems of equations since about 1750. Leibnitz, a great German mathematician, first discovered the notation and wrote about it to a friend in 1693, but its general use did not come until much later. To-day determinants are extensively used in all fields of advanced mathematics. For a fuller treatment of determinants, see the chapter on determinants in the STONE-MILLIS HIGHER ALGEBRA.

EXERCISES

Solve by determinants:

1. $\begin{cases} 4x + y = 6, \\ x + y = 3. \end{cases}$

2. $\begin{cases} 3M + 2N = 12, \\ 2M + 3N = 13. \end{cases}$

3. $\begin{cases} p + 5q = 8, \\ 2p - q = 5. \end{cases}$

4. $\begin{cases} 8a - b = 4, \\ 6a + 2b = 14. \end{cases}$

5. $\begin{cases} R - T = 1, \\ 2R + T = 14. \end{cases}$

6. $\begin{cases} 12v - t = 10, \\ 5v + 7t = 16. \end{cases}$

7. $\begin{cases} -2x + 5y = 4, \\ 7x - y = 5. \end{cases}$

8. $\begin{cases} 4h = 5 - k, \\ h + 3 = 2k. \end{cases}$

9. $\begin{cases} 2P - 1 = Q, \\ P + 5Q = 9. \end{cases}$

10. $\begin{cases} 9m - 4n = 20, \\ 5m - n = 16. \end{cases}$

11. $\begin{cases} 2V + 1 = 5a, \\ 3a - 4 = 6V. \end{cases}$

12. $\begin{cases} 3 - n = 2m, \\ 4m + 2 = n. \end{cases}$

60.* Determinants of the Third Order. — Solving the system

$$\begin{cases} ax + by + cz = p, \\ dx + ey + fz = q, \\ gx + hy + iz = r, \end{cases}$$

$$x = \frac{pei + qhc + rfb - cer - fhp - iqb}{aei + dhc + gfb - ceg - fha - idb},$$

$$y = \frac{aqi + drc + gfp - cqg - fra - idp}{aei + dhc + gfb - ceg - fha - idb},$$

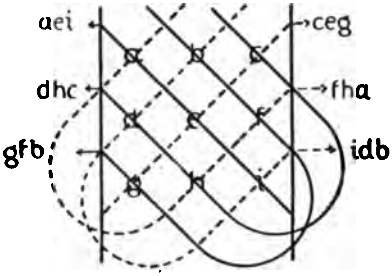
$$z = \frac{aer + dhp + gqb - peg - qha - rdb}{aei + dhc + gfb - ceg - fha - idb}.$$

The denominator $aei + dhc + gfb - ceg - fha - idb$ of each fraction may be written in the form

$$\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix}.$$

This symbol is called a **determinant of the third order.**

It is seen that each term of the denominator is the product of three numbers, and that these three numbers are chosen so that *one and only one is taken from each row and one and only one from each column* of the determinant. It is seen also that if the numbers of the determinant are connected by lines, as in the accompanying figure, *the three numbers on each of the three continuous lines form one of the positive terms, and the three numbers on each of the dotted lines form one of the negative terms.*



These facts will enable one to write down the value of any given determinant of the third order.

EXAMPLE. — $\begin{vmatrix}2 & 6 & 5\\1 & 4 & 3\\8 & 1 & 7\end{vmatrix} = 2\times4\times7 + 1\times1\times5 + 8\times6\times3 - 8\times4\times5 - 1\times3\times2 - 7\times1\times6 = -3.$

EXERCISES

Find the value of:

1. $\begin{vmatrix}2 & 3 & 5\\1 & 2 & 1\\4 & 1 & 3\end{vmatrix}$.

2. $\begin{vmatrix}1 & 6 & 2\\7 & 8 & 1\\1 & 3 & 4\end{vmatrix}$.

3. $\begin{vmatrix}4 & 8 & 1\\3 & 1 & 5\\6 & 1 & 2\end{vmatrix}$.

4. $\begin{vmatrix}1 & 2 & 3\\2 & 1 & 1\\3 & 2 & 1\end{vmatrix}$.

5. $\begin{vmatrix}1 & 1 & 1\\4 & 4 & 4\\1 & 5 & 2\end{vmatrix}$.

6. $\begin{vmatrix}8 & 2 & -3\\1 & -1 & 4\\2 & -2 & 1\end{vmatrix}$.

7. $\begin{vmatrix}3 & -5 & -1\\4 & -2 & 3\\1 & -1 & 1\end{vmatrix}$.

8. $\begin{vmatrix}12 & 1 & -1\\2 & -4 & 8\\1 & -2 & -1\end{vmatrix}$.

9. $\begin{vmatrix}10 & 0 & 4\\3 & 6 & -2\\0 & -1 & 5\end{vmatrix}$.

10. $\begin{vmatrix}a & b & c\\1 & 1 & 1\\1 & 2 & 3\end{vmatrix}$.

11. $\begin{vmatrix}x & y & z\\y & x & z\\1 & 1 & 1\end{vmatrix}$.

12. $\begin{vmatrix}1 & 0 & n\\0 & 1 & 0\\n & 1 & 1\end{vmatrix}$.

61.* Systems of Three Linear Equations Solved by Determinants. — It follows from the definition of a determinant of the third order in § 60 that the solution of the system

$$\begin{cases} ax + by + cz = p, \\ dx + ey + fz = q, \\ gx + hy + iz = r, \end{cases}$$

may be written in the form

$$x = \frac{\begin{vmatrix} p & b & c \\ q & e & f \\ r & h & i \end{vmatrix}}{\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix}}, \quad y = \frac{\begin{vmatrix} a & p & c \\ d & q & f \\ g & r & i \end{vmatrix}}{\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix}}, \quad z = \frac{\begin{vmatrix} a & b & p \\ d & e & q \\ g & h & r \end{vmatrix}}{\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix}}.$$

It is seen that the numerator of x is obtained from the denominator by replacing the coefficients of x that form the first column by the known terms of the equations, and that the numerators of y and z are obtained in a similar manner.

Hence, any system of three linear equations in three unknown numbers may be solved by merely writing down the solution in the form of determinants, and simplifying.

EXAMPLE. — Solve $\begin{cases} x - 2y + z = 6, \\ x + 3y + 2z = 13, \\ 2x - y + z = 13. \end{cases}$

$$x = \frac{\begin{vmatrix} 6 & -2 & 1 \\ 13 & 3 & 2 \\ 13 & -1 & 1 \end{vmatrix}}{\begin{vmatrix} 1 & -2 & 1 \\ 1 & 3 & 2 \\ 2 & -1 & 1 \end{vmatrix}} = \frac{18 - 13 - 52 - 39 + 26 + 12}{3 - 1 - 8 - 6 + 2 + 2} = \frac{-48}{-8} = 6.$$

Similarly, $y = 1$, and $z = 2$.

EXERCISES

Solve by determinants:

1. $$\begin{cases} x+y+z=7, \\ 2x+y-z=7, \\ 3x-y+2z=15. \end{cases}$$

2. $$\begin{cases} A-2B+C=2, \\ 4A+B-2C=20, \\ A+3B-5C=6. \end{cases}$$

3. $$\begin{cases} 3p+2q+r=5, \\ 2p-3q-r=1, \\ 5p-q+2r=14. \end{cases}$$

4. $$\begin{cases} M+2N-P=5, \\ 6M+5N+P=2, \\ 2M-N-8P=1. \end{cases}$$

5. $$\begin{cases} 3a+5b+c=1, \\ a-4b-2c=32, \\ 7a+5b-4c=36. \end{cases}$$

6. $$\begin{cases} 9r-s+2t=8, \\ r+3s-t=5, \\ 4r-s-3t=1. \end{cases}$$

7. $$\begin{cases} x-2=y+z, \\ 3x+y=z-1, \\ x+z=4-y. \end{cases}$$

8. $$\begin{cases} 12m-3n=4-2k, \\ m+k=2-n, \\ 5k-4n=2m+1. \end{cases}$$

9. $$\begin{cases} A+B=10, \\ B+C=6, \\ A+C=8. \end{cases}$$

10. $$\begin{cases} x-y=2, \\ y-z=3, \\ x-z=5. \end{cases}$$

SUPPLEMENTARY EXERCISES

Solve:

1. $$\begin{cases} \dfrac{1}{8a}+\dfrac{7}{4b}=\dfrac{5}{16}, \\ \dfrac{1}{4a}-\dfrac{3}{2b}+\dfrac{5}{65}=0. \end{cases}$$

2. $$\begin{cases} \dfrac{2}{5m}+\dfrac{5}{2n}=7, \\ \dfrac{5}{2m}+\dfrac{1}{n}=14\tfrac{1}{2}. \end{cases}$$

3. $$\begin{cases} \dfrac{1}{x+1}+\dfrac{1}{y+1}=\dfrac{7}{12}, \\ \dfrac{1}{x+1}-\dfrac{1}{y+1}=\dfrac{1}{12}. \end{cases}$$

4. $$\begin{cases} \dfrac{2}{a-2}+\dfrac{1}{b-2}=\dfrac{13}{15}, \\ \dfrac{1}{a-2}+\dfrac{2}{b-2}=\dfrac{11}{15}. \end{cases}$$

5. $$\begin{cases} \dfrac{8}{2m+3}-\dfrac{6}{2n-3}=5, \\ \dfrac{2}{2m+3}+\dfrac{5}{2n-3}+2=0 \end{cases}$$

6. $$\begin{cases} \dfrac{1}{A-5}+\dfrac{2}{B-5}+\dfrac{1}{C-5}=2, \\ \dfrac{4}{A-5}-\dfrac{8}{B-5}+\dfrac{3}{C-5}=3, \\ \dfrac{3}{A-5}-\dfrac{1}{B-5}-\dfrac{1}{C-5}=\dfrac{1}{4}. \end{cases}$$

7. $$\begin{cases} \dfrac{2}{3x}+\dfrac{5}{2y}+\dfrac{1}{z}=3\tfrac{5}{6}, \\ \dfrac{1}{x}-\dfrac{1}{3y}+\dfrac{2}{z}=2\tfrac{1}{6}, \\ \dfrac{1}{2x}-\dfrac{2}{3y}-\dfrac{1}{2z}=-\dfrac{11}{12}. \end{cases}$$

8. In the equations $\frac{P}{R}=\frac{a}{a'}$ and $\frac{R}{Q}=\frac{b}{b'}$, eliminate R by multiplication.

9. In the formula $V=\frac{1}{6}H(B+b+4M)$ and $4M=B+b+2\sqrt{Bb}$, eliminate M, and thus derive a formula between V, H, B, and b.

10. In the formula $P_1d_1=P_2d_2$ and $P_1+P_2=w$ solve for P_1 and P_2.

Solve each of the following systems for x, y, and z:

11. $\begin{cases} x+y+z=a, \\ x+y-z=b, \\ x-y+z=c. \end{cases}$

12. $\begin{cases} ax+by=p, \\ cy+dz=q, \\ ex+fz=r. \end{cases}$

13. $\begin{cases} \frac{1}{x}+\frac{1}{y}=4, \\ \frac{1}{x}+\frac{1}{z}=3, \\ \frac{1}{y}+\frac{1}{z}=2. \end{cases}$

14. $\begin{cases} \frac{1}{x}+\frac{1}{y}=a, \\ \frac{1}{x}+\frac{1}{z}=b, \\ \frac{1}{y}+\frac{1}{z}=c. \end{cases}$

CHAPTER VI

ROOTS, SURDS, AND IMAGINARY NUMBERS

62. Roots. — The numbers whose square, cube, fourth power, fifth power, etc., are a given number are called the **square root, cube root, fourth root, fifth root,** etc., respectively, of the given number. In general, the number whose nth power is a given number is the **nth root** of that number.

Thus, since $(4x^2)^3 = 64x^6$, $4x^2$ is the cube root of $64x^6$.

A root of a number is indicated by placing before it the sign $\sqrt{\ }$, called the **radical sign.** Usually a vinculum is placed over the number to show how far the effect of the radical sign is to extend. Except in case of a square root, a little number called the **index** of the root is written above the radical sign to indicate what root it is. In the case of square root the index 2 is understood.

Thus, the square root of x is written $\sqrt{x}$; the cube root of n^2+1 is written $\sqrt[3]{n^2+1}$; the fourth root of a^2-b^2 is written $\sqrt[4]{a^2-b^2}$; etc.

63. Number of Roots of a Number. — Since $(+2)^2 = 4$ and $(-2)^2 = 4$, $\sqrt{4}$ is either $+2$ or -2, usually written $\sqrt{4} = \pm 2$. Similarly, since $(+3)^4 = 81$, and $(-3)^4 = 81$, $\sqrt[4]{81}$ may be either $+3$ or -3. In general,

Every positive number has two even roots that may be expressed as positive or negative numbers, these roots having the same absolute value, but opposite signs.

Since no positive or negative number raised to an even power can give a negative number, *A negative number has no even root that can be expressed as a positive or negative number.*

Since $(+2)^3 = 8$ and $(-2)^3 = -8$, $\sqrt[3]{8} = +2$ and $\sqrt[3]{-8} = -2$. Similarly, since $(+2)^5 = 32$ and $(-2)^5 = -32$, $\sqrt[5]{32} = +2$ and $\sqrt[5]{-32} = -2$. In general,

Any given number has one odd root that has the same sign as the given number.

The *indicated even root* of a negative number is called an **imaginary number**. All other numbers are called **real numbers**, for distinction.

Thus, $\sqrt{-9}$ and $\sqrt[4]{-1}$ are *imaginary numbers*, and 3 and $\sqrt{5}$ are *real numbers*.

It can be shown that every number has *two* square roots, *three* cube roots, *four* fourth roots, etc., expressed in terms of real and imaginary numbers.

Thus, the three cube roots of 8 are 2, $-1 + \sqrt{-3}$, and $-1 - \sqrt{-3}$. For the cube of each of these is 8. The four fourth roots of 81 are 3, -3, $\sqrt{-9}$, and $-\sqrt{-9}$. For the fourth power of each of these is 81. The six sixth roots of 64 are 2, -2, $1 + \sqrt{-3}$, $1 - \sqrt{-3}$, $-1 + \sqrt{-3}$, and $-1 - \sqrt{-3}$.

64. The Principal Root. — The positive real even root of a positive number, or the real odd root of a positive or negative number, is called the **principal root**.

Thus, the principal square root of 9 is 3, the principal cube root of 64 is 4, the principal cube root of -27, is -3, etc.

Unless otherwise stated, only the principal root of a number is to be understood as indicated in an expression.

Thus, in $1 + \sqrt{3}$ only the principal square root of 3 is understood. Similarly, in finding the values of m and n that satisfy the equation $\sqrt{m} - \sqrt{n} = 1$, only the principal square roots of m and n are allowed to be used.

65. Root of a Monomial. — As shown in the FIRST COURSE,

To find the principal square root of a monomial take the square root of the numerical coefficient, and divide the exponent of each literal factor by 2.

To find the principal cube root of a monomial, take the cube root of the numerical coefficient, and divide the exponent of each literal factor by 3.

Thus, since $(3m^4n^3)^2 = 9m^8n^6$, the principal square root of $9m^8n^6$ is $3m^4n^3$, which may be obtained by the above rule.

Since $(2x^2y^4)^3 = 8x^6y^{12}$, the principal cube root of $8x^6y^{12}$ is $2x^2y^4$, which may be obtained by the above rule.

Similarly, in general,

To find the principal nth root of a monomial, take the nth root of the numerical coefficient, and divide the exponent of each literal factor by n.

EXERCISES

Find the principal square roots of:

1. 25. 3. n^6. 5. $16a^2b^4$. 7. $100m^4n^8$. 9. $4p^6q^2r^4$.
2. 81. 4. P^{10}. 6. $36V^4t^6$. 8. $121A^2B^{12}$. 10. $9x^4y^6z^8$.

Find the principal cube roots of:

11. 27. 13. A^6. 15. $8x^3y^6$. 17. $125a^{12}b^{15}$. 19. $8x^3y^6z^9$.
12. 64. 14. s^{12}. 16. $27P^6Q^9$. 18. $64W^9V^{18}$. 20. $27m^{12}n^3p^{21}$.

Find the principal fourth roots of:

21. 16. 22. 81. 23. a^8b^{12}. 24. $256x^4y^{16}$. 25. $625M^8N^{12}P^4$.

Find the principal fifth roots of:

26. 32. 27. 243. 28. x^5y^{10}. 29. $32A^{15}B^{20}$. 30. $p^{25}q^{15}r^{30}s^{20}$.

66. Square Root of a Polynomial. — As shown in the FIRST COURSE (see § 135), to find the principal square root of a polynomial:

(1) *Write the given polynomial in descending or ascending powers of some letter.*

(2) *Take the square root of the first term for the first term of the root, and subtract the first term from the polynomial.*

(3) *Double the root found for the first trial divisor, divide the first term of the remainder by this, and write the quotient as the second term of the root.*

(4) *Add this quotient to the trial divisor to obtain the true divisor, multiply the true divisor by the second term of the root, and subtract the product from the preceding remainder.*

(5) *If there is still a remainder, double all of the root already found, for a new trial divisor, and proceed as before. Continue this process until all terms of the root are found.*

Example. — Find the square root of $a^4 - 10a^3 + 29a^2 - 20a + 4$.

$$\begin{array}{l|l}
 & a^4 - 10a^3 + 29a^2 - 20a + 4 \;\big|\; \underline{a^2 - 5a + 2} \\
 & \underline{a^4 \qquad\qquad\qquad\qquad\qquad\qquad} \\
2a^2 & -10a^3 + 29a^2 - 20a + 4 \\
2a^2 - 5a & \underline{-10a^3 + 25a^2 \qquad\qquad\quad} \\
2a^2 - 10a & \qquad\qquad 4a^2 - 20a + 4 \\
2a^2 - 10a + 2 & \qquad\qquad \underline{4a^2 - 20a + 4}
\end{array}$$

The square root is $a^2 - 5a + 2$.

EXERCISES

Find the square root of:

1. $A^2 - 6A + 9$.
2. $4t^2 + 12t + 9$.
3. $1 - 10V + 25V^2$.
4. $49m^4 + 28m^2f + 4f^2$.
5. $36a^2 - 48ab + 16b^2$.
6. $16xy + y^2 + 64x^2$.
7. $n^4 - 6n^3 + 13n^2 - 12n + 4$.
8. $1 + 10k + 29k^2 + 20k^3 + 4k^4$.
9. $t^6 - 4t^3 + 25t^2 + 4 + 20t - 10t^4$.
10. $16R^4 + 30R^3 - 24R^2 + 9 + 25R^6 - 40R^5$.
11. $a^4 - 2a^3b + 3a^2b^2 - 2ab^3 + b^4$.
12. $36a^2 + 12ab + b^2 - 24ac - 4bc + 4c^2$.

13. $x^2 + 25y^2 + 6xz - 10xy - 30yz + 9z^2$.

14. $v^6 - 4v^5 - 24v^3 + 33v^2 + 14v^4 - 20v + 4$.

Find the first four terms of the square root of:

15. $1 - 4x$. 16. $1 + 8y$. 17. $1 - a$.

Find the fourth root of:

18. $x^4 - 12x^3 + 54x^2 - 108x + 81$.

SUGGESTION. — Find the square root of the square root.

19. $m^4 - 8m^3n + 24m^2n^2 - 32mn^3 + 16n^4$.

67. Square Root of an Arithmetical Number. — An arithmetical number is in nature a polynomial. Hence a process similar to that in § 66 may be used for finding the square root of an arithmetical number.

It is easily seen that, if the figures of an integer are marked off from right to left into groups of two, the number of figures in the integral part of the square root will be equal to the number of groups, any one figure that remains on the left being counted as a whole group.

Thus, marked off into groups, 42579 becomes 4′25′79′. Hence the integral part of the square root will contain three figures.

The similarity of the process of finding the square root of an integer to that of § 66 is shown in the following example:

EXAMPLE. — Find the square root of 119,716.

Pointing off gives 11′97′16. Hence the square root has three figures — hundreds, tens, and ones. The root lies between 300 and 400, because 300^2 is less and 400^2 more than 119,716. That is, the value of the hundreds in the root is 300.

The work is as follows:

$$\begin{array}{r|r|l}
 & 11'97'16' & \underline{300 + 40 + 6 = 346} \\
 & 9\ 00\ 00 & \\
\hline
2 \times 300 = 600 & 2\ 97\ 16 & \\
600 + 40 = 640 & 2\ 56\ 00 & \\
\hline
2 \times 340 = 680 & 41\ 16 & \\
680 + 6 = 686 & 41\ 16 & \\
\hline
\end{array}$$

Explanation. — $300^2 = 90000$. Subtracting this leaves 29716. Doubling 300 gives 600 for the first trial divisor. Dividing 29716 by 600 gives approximately 40, the value of the tens in the root. Adding 40 to 600 gives 640, the first true divisor. Multiplying 640 by 40 gives 25600. Subtracting this leaves 4116. Doubling 340, the value of the part of the root found, gives 680 for the second trial divisor. Dividing 4116 by 680 gives approximately 6. Adding 6 to 680 gives 686, the second true divisor. Multiplying 686 by 6 gives 4116. Subtracting this leaves no remainder.

Hence the exact root is $300 + 40 + 6$ or 346.

If a number that contains a decimal is marked off into groups of two figures each, beginning at the decimal point and marking both to the left and right, the number of figures in the integral and decimal parts of the square root will equal the number of periods in the integral and decimal parts, respectively, of the given number. If the decimal part has an odd number of figures, add a zero to make it even.

To find the approximate value, to any number of decimal places, of the square root of a number that is not a perfect square, mark the decimal point, annex zeros, and proceed as with a decimal.

Example. — Find to two decimal places the square root of 8.

We annex 4 zeros after the decimal point.

$$\begin{array}{r|l}
 & 8.'00'00' \;\big|\; \underline{2 + .8 + .02 = 2.82} \\
 & 4.\;00\;00 \\
2 \times 2 = 4 & 4.\;00\;00 \\
4 + .8 = 4.8 & \underline{3.\;84\;00} \\
2 \times 2.8 = 5.6 & .\;16\;00 \\
5.6 + .02 = 5.62 & \underline{.\;11\;24}
\end{array}$$

EXERCISES

Find the square root of:

1. 2209.
2. 8464.
3. 131044
4. 649636.
5. 2673225
6. 68.89.
7. 1.6129.
8. 0.1849.
9. 0.007396.

Find to three decimal places the square root of:

10. 12.	12. 20.	14. 1.61.
11. 7.	13. 0.2.	15. 0.05.

16. The legs of a right triangle are 48 inches and 64 inches. Find the hypotenuse.

17. The hypotenuse of a right triangle is 14 inches and one leg 8 inches. Find the length of the other leg.

18. A rectangle is 19 inches long and 11 inches wide. What is the length of the diagonal?

19. Find the radius of a circle whose area is 60 square inches.

20. The boiler of an engine contains 96 flues or pipes, each 3 inches in diameter. If the area of the cross section of the smokestack equals the combined areas of the cross sections of all of the flues, what must be the diameter of the smokestack?

21. Three drainage pipes, each 6 inches in diameter, empty into one larger pipe. If the carrying capacity of the larger pipe is equal to the sum of the carrying capacities of the smaller pipes, what must be its diameter?

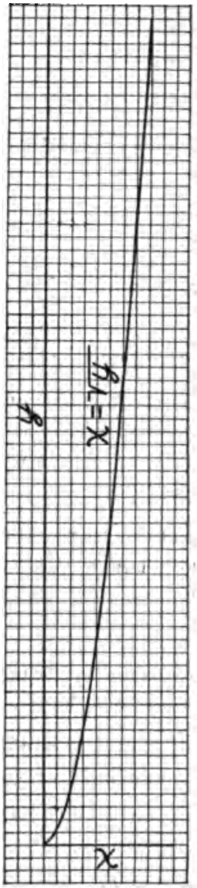
SUGGESTION. — Consider that the area of a cross section of the larger pipe must equal the sum of the cross sectional areas of the three smaller pipes.

22. The area of a triangle whose sides are a, b, and c equals $\sqrt{s(s-a)(s-b)(s-c)}$, where s is one half of the sum of the three sides. Find the area of a triangle whose sides are 12 inches, 16 inches, and 18 inches.

23. Find the area of a triangle whose sides are 46 feet, 30 feet, and 24 feet.

68. Roots of Numbers found Graphically. — If y represents any number and x the square root of that number, then $x=\sqrt{y}$. By assigning values to y and computing the corresponding values of x, we get the following sets of values of x and y that satisfy this equation:

y	0	1	4	9	16	25	36	49	64
x	0	1	2	3	4	5	6	7	8



Locating the points corresponding to these sets of values of x and y, we get the graph in the margin.

By means of this graph we can read off the square root of any number between 0 and 64, approximated to tenths.

Note. — If desired, the student may make such a graph on a larger piece of squared paper and with a larger scale, say 1 space to the unit along the y-axis and 5 spaces to the unit along the x-axis. The larger graph could be used for getting the roots of larger numbers, and for getting the roots with greater precision.

Similar graphs may be constructed for finding the cube roots, fourth roots, etc., of numbers.

EXERCISES

Find to tenths, by use of the above graph, the square roots of:

1. 10. 3. 20. 5. 40. 7. 60. 9. 2.5.
2. 15. 4. 28. 6. 52. 8. 6.5. 10. 12.5.

11. By constructing the graph of $x = \sqrt[3]{y}$, a curve is obtained for finding the cube roots of numbers graphically. By assigning to y the values 0, 1, 8, 27, 64, 125, 216, get the corresponding values of x, and draw the graph.

12. Use the graph in Ex. 11 for finding to tenths the cube roots of 12, 20, 45, 100, and 150.

69. Surds. — An indicated root of any number is called a **radical**. Thus, $\sqrt{4}$, $\sqrt[3]{10}$, $\sqrt{x^2 - y^2}$, are radicals.

Any expression containing one or more radicals is called a **radical expression.**

If the exact value of a radical which is real cannot be computed, that is, cannot be exactly expressed by a whole number or common fraction, it is called a **surd.**

Thus, $\sqrt{3}$, $\sqrt[3]{12}$, $\sqrt{1-x^2}$ are surds.

The index of a surd indicates the **order** of the surd. A surd whose index is 2 is a surd of the **second order,** or a **quadratic surd.** A surd whose index is 3 is of the **third order.** In general, a surd whose index is n is of the **nth order.**

Thus, $\sqrt{5}$ is of the *second order;* $\sqrt[3]{2x}$ of the *third order;* $\sqrt[4]{a+b}$ of the *fourth order;* etc.

NOTE. — Although the exact value of a quadratic surd cannot be computed, it can be represented graphically by a line of definite length. See FIRST COURSE, § 143.

70. Simplest Form of a Surd. — The *form* of a surd may sometimes be changed without altering its value.

If a surd contains under the radical sign a factor of which the indicated root can be extracted, this factor may be removed from under the radical.

Thus, $\sqrt[3]{a^6b}=a^2\sqrt[3]{b}$, and $\sqrt{48}=4\sqrt{3}$.

If a surd contains a fraction under the radical sign, this fraction may be removed.

Thus, $\sqrt{\frac{2}{3}}=\frac{1}{3}\sqrt{6}$, and $\sqrt[4]{\frac{m}{n^3}}=\frac{1}{n}\sqrt[4]{mn}$.

If the index of a surd and the exponents of all factors under the radical sign have a common divisor, the surd may be changed to lower order.

Thus, $\sqrt[6]{a^2b^4}=\sqrt[3]{ab^2}$, and $\sqrt[4]{9}=\sqrt{3}$.

Removing a factor or fraction from under the radical sign, or reducing the order of a surd, is called **simplifying the surd.**

A surd is in its **simplest form** when the expression under the radical sign is integral, contains no factor of which the indicated root can be extracted, and the exponents of its factors and the index have no common divisor.

71. Simplifying a Surd. — To simplify a surd by removing from under the radical sign a factor of which the indicated root can be extracted, use is made of the following principle:

The nth root of a product equals the product of the nth roots of its factors; that is,

$$\sqrt[n]{ab} = \sqrt[n]{a}\,\sqrt[n]{b}.$$

This principle is true because $(\sqrt[n]{a}\,\sqrt[n]{b})^n = (\sqrt[n]{a})^n(\sqrt[n]{b})^n = ab.$

Example. — Simplify $\sqrt[4]{32\,a^4b^8c^7}$.

Separating $32\,a^4b^8c^7$ into two factors so that the fourth root of one can be extracted,

$$\sqrt[4]{32\,a^4b^8c^7} = \sqrt[4]{(16\,a^4b^8c^4)(2\,c^3)} = \sqrt[4]{16\,a^4b^8c^4}\,\sqrt[4]{2\,c^3} = 2\,ab^2c\sqrt[4]{2\,c^3}.$$

To simplify a surd containing a fraction under the radical sign, first multiply both terms of the fraction by such an expression as will give a new denominator of which the indicated root can be extracted, then write the new fraction as the product of two factors, and apply the above principle.

Example 1. — Simplify $\sqrt[3]{\frac{16}{25}}$.

Multiplying both terms of the fraction by 5,

$$\sqrt[3]{\frac{16}{25}} = \sqrt[3]{\frac{80}{125}} = \sqrt[3]{\frac{8}{125}\times 10} = \sqrt[3]{\frac{8}{125}}\;\sqrt[3]{10} = \tfrac{2}{5}\sqrt[3]{10}.$$

Example 2. — Simplify $\sqrt[4]{\frac{5}{2\,n}}$.

$$\sqrt[4]{\frac{5}{2\,n}} = \sqrt[4]{\frac{40\,n^3}{16\,n^4}} = \sqrt[4]{\frac{1}{16\,n^4}\times 40\,n^3} = \sqrt[4]{\frac{1}{16\,n^4}}\;\sqrt[4]{40\,n^3} = \frac{1}{2\,n}\sqrt[4]{40\,n^3}.$$

To simplify a surd by reducing it to lower order, use is made of the following principle:

The value of a surd is not changed by dividing the index of the surd and the exponents of all factors under the radical sign by the same number. That is,

$$\sqrt[mx]{a^{nx}b^{px}} = \sqrt[m]{a^nb^p}.$$

To prove this principle, let $\sqrt[mx]{a^{nx}b^{px}} = r$.

Then $a^{nx}b^{px} = r^{mx}$, by definition of a root.

Hence, $\sqrt[x]{a^{nx}b^{px}} = \sqrt[x]{r^{mx}}$, or, $a^nb^p = r^m$(§ 65).

Hence, $\sqrt[m]{a^nb^p} = \sqrt[m]{r^m} = r$.

Therefore, $\sqrt[mx]{a^{nx}b^{px}} = \sqrt[m]{a^nb^p}$, because each equals r.

EXAMPLE 1. — Simplify $\sqrt[12]{a^4b^8}$.

Dividing index and exponents by 4,

$$\sqrt[12]{a^4b^8} = \sqrt{ab^2}.$$

EXAMPLE 2. — Simplify $\sqrt[6]{125}$.

$$\sqrt[6]{125} = \sqrt[6]{5^3} = \sqrt{5}.$$

A surd sometimes may be simplified by applying both of the above principles.

EXAMPLE. — Simplify $\sqrt[4]{144\,x^6y^8}$.

$$\sqrt[4]{144\,x^6y^8} = \sqrt[4]{12^2\,x^6y^8} = \sqrt{12\,x^3y^4} = \sqrt{(4\,x^2y^4)(3\,x)} = 2\,xy^2\sqrt{3\,x}.$$

EXERCISES

Reduce to the simplest form:

1. $\sqrt{8}$.
2. $\sqrt{27}$.
3. $\sqrt{12}$.
4. $\sqrt{72}$.
5. $\sqrt[3]{16}$.
6. $\sqrt[3]{24}$.
7. $\sqrt[3]{72}$.
8. $\sqrt[3]{250}$.
9. $\sqrt[4]{32}$.
10. $\sqrt[4]{162}$.
11. $\sqrt[5]{64}$.
12. $\sqrt[5]{224}$.
13. $\sqrt{\frac{1}{8}}$.
14. $\sqrt{\frac{2}{5}}$.
15. $\sqrt{\frac{8}{27}}$.
16. $\sqrt[3]{\frac{1}{2}}$.
17. $\sqrt[3]{\frac{3}{4}}$.
18. $\sqrt[3]{\frac{5}{7}}$.
19. $\sqrt[4]{\frac{1}{4}}$.
20. $\sqrt[5]{\frac{3}{16}}$.

21. $\sqrt{a^3b^2}$.

22. $\sqrt{3\,m^5n}$.

23. $\sqrt{8\,x^2y^7}$.

24. $\sqrt[3]{5\,v^4t}$.

25. $\sqrt[3]{16\,R^5r^4}$.

26. $\sqrt[4]{32\,A^2B^5}$.

27. $\sqrt{80\,x^4y^2}$.

28. $\sqrt[4]{a^2b^2}$.

29. $\sqrt[6]{a^4b^8}$.

30. $\sqrt[8]{16\,x^{12}y^8}$.

31. $\sqrt[6]{27\,w^3v^{12}}$.

32. $\sqrt[10]{32\,P^5Q^{15}}$.

33. $\sqrt[12]{16\,x^8y^4z^{12}}$.

34. $\sqrt{\frac{a}{b}}$.

35. $\sqrt{\frac{x}{y^3}}$.

36. $\sqrt{\frac{3\,m}{2\,n}}$.

37. $\sqrt[4]{\frac{1}{4\,x^2}}$.

38. $\sqrt[3]{\frac{a^2b}{5\,c^4}}$.

39. $\sqrt[3]{\frac{16\,x^5}{27\,y^2}}$.

40. $\sqrt[4]{\frac{25\,v^6}{49\,t^2}}$.

41. $\sqrt{\frac{a+b}{a-b}}$.

42. $\sqrt{\frac{m^2n}{m+n}}$.

43. $\sqrt{\frac{B^2}{4\,A^2}-\frac{C}{A}}$.

44. $\sqrt{\frac{x}{y^3}+\frac{y}{x^3}}$.

45. $\frac{3\,x}{2\,y}\sqrt[3]{\frac{27\,y^4}{x^4}}$.

46. $\frac{a^2b}{2}\sqrt[5]{\frac{a^6b^2}{27}}$.

47. $\frac{1}{K^2}\sqrt[4]{\frac{1}{K^2}-\frac{2}{K}+1}$.

48. $\sqrt{4-8\sqrt{3}}$.

49. $\sqrt{R^2-3\,R^2\sqrt{5}}$.

50. $\sqrt{(a-b)(a^2-b^2)}$.

72. Addition and Subtraction of Surds. — Two terms containing the same surd factor may be added or subtracted by adding or subtracting the coefficients of the surd factor.

Example. — $4\sqrt{2}+3\sqrt{2}=7\sqrt{2}$.

Terms containing surd factors that are not of the same form may sometimes be added or subtracted by first reducing the surd factors to the same form. In such cases, all of the surds should be reduced to the simplest form.

Example. — Simplify $\sqrt{32}+\sqrt{50}-\sqrt{8}$.

$$\sqrt{32}+\sqrt{50}-\sqrt{8}=4\sqrt{2}+5\sqrt{2}-2\sqrt{2}=7\sqrt{2}.$$

EXERCISES

Simplify:

1. $\sqrt{18}-\sqrt{8}+\sqrt{32}$.
2. $\sqrt{75}-\sqrt{3}-\sqrt{12}$.
3. $\sqrt{20}+\sqrt{45}-\sqrt{80}$.
4. $\sqrt{294}+\sqrt{486}-\sqrt{24}$.
5. $\sqrt[3]{16}-\sqrt[3]{54}+2\sqrt[3]{2}$.
6. $2\sqrt[3]{3}+\sqrt[3]{24}+5\sqrt[3]{192}$.
7. $\sqrt{\frac{1}{2}}+\sqrt{\frac{1}{8}}$.
8. $\sqrt{\frac{2}{3}}-\sqrt{\frac{3}{8}}$.
9. $4\sqrt[3]{\frac{2}{9}}-\sqrt[3]{\frac{3}{4}}$.
10. $9\sqrt[3]{\frac{4}{9}}+\sqrt[3]{\frac{3}{2}}$.
11. $\sqrt{32}+\sqrt[4]{324}$.
12. $\sqrt{75}-\sqrt[6]{1728}$.
13. $\sqrt{4n}+\sqrt{9n}+\sqrt{16n}-\sqrt{25n}$.
14. $\sqrt[3]{8t^2}-\sqrt[3]{125t^2}-\sqrt[3]{t^2}+6\sqrt[3]{64t^2}$.
15. $\sqrt{a^3}-2a\sqrt[4]{a^2}+3a\sqrt[6]{a^3}-\sqrt[8]{a^{12}}$.
16. $5m\sqrt[6]{(m-n)^3}+\sqrt{(m-n)^3}+n\sqrt[4]{(m-n)^2}$.
17. $3\sqrt{4m^3-4m^2n}-\sqrt{mn^2-n^3}+\sqrt{(m^2-n^2)(m+n)}$.
18. $\sqrt[3]{(A+B)^4}+(A-B)\sqrt[3]{A+B}+A\sqrt[6]{A^2+2AB+B^2}$.
19. $\sqrt{\frac{5x}{y}}+\sqrt{\frac{5y}{x}}+\left(\frac{1}{x}-\frac{1}{y}\right)\sqrt{5xy}$.
20. $(p-q)\sqrt{\frac{p+q}{q-q}}+\frac{p+q}{q}\sqrt{\frac{pq^2-q^3}{p+q}}+8\sqrt{p^2-q^2}$.

73. Changing Surds to the Same Order. — It follows from the second principle in § 71 that

The value of a surd is not changed by multiplying the index of the surd and the exponents of the factors under the radical sign by the same number. That is,

$$\sqrt[m]{a^n b^p}=\sqrt[mx]{a^{nx}b^{px}}.$$

By use of this principle surds of different orders may be changed to surds of the same order, without altering their values. The lowest order to which two or more surds may be changed is the least common multiple of their given orders.

Example 1. — Reduce to the same order $\sqrt[4]{v^2t^3}$ and $\sqrt[6]{v^3t^2}$.

The l. c. m. of the indices is 12. Multiplying index and exponents of $\sqrt[4]{v^2t^3}$ by 3 and index and exponents of $\sqrt[6]{v^3t^2}$ by 2,

$$\sqrt[4]{v^2t^3}=\sqrt[12]{v^6t^9} \text{ and } \sqrt[6]{v^3t^2}=\sqrt[12]{v^6t^4}.$$

Example 2. — Reduce to the same order $\sqrt{2}$ and $\sqrt[5]{3}$.

$$\sqrt{2}=\sqrt[10]{2^5}=\sqrt[10]{32} \text{ and } \sqrt[5]{3}=\sqrt[10]{3^2}=\sqrt[10]{9}.$$

EXERCISES

Reduce to the same order:

1. $\sqrt[3]{n^2}$, $\sqrt[4]{n^3}$, $\sqrt[6]{n^5}$.
2. $\sqrt{V}$, $\sqrt[4]{V^2}$, $\sqrt[5]{V^3}$.
3. $\sqrt{2x}$, $\sqrt[3]{3x}$, $\sqrt[3]{2x^2}$.
4. $\sqrt{3t^3}$, $\sqrt[3]{2t^2}$, $\sqrt[4]{t}$.
5. $\sqrt{a^5b^3}$, $\sqrt[5]{a^4b}$, $\sqrt[4]{ab^3}$.
6. $\sqrt{2}$, $\sqrt[3]{3}$.
7. $\sqrt[3]{4}$, $\sqrt[4]{6}$, $\sqrt{2}$.
8. $\sqrt[5]{3N^3}$, $\sqrt[6]{2N^2}$.
9. $\sqrt{\frac{1}{2}}$, $\sqrt[3]{\frac{1}{3}}$, $\sqrt[4]{\frac{1}{4}}$.
10. $\sqrt[3]{5y^2z}$, $\sqrt{\frac{1}{2}yz^2}$, $\sqrt[4]{\frac{1}{8}y^3}$.

Determine which is the greater:

11. $\sqrt[3]{16}$ or $\sqrt{6}$.

Suggestion. — Change them to the same order.

12. $\sqrt[3]{11}$ or $\sqrt{5}$. 13. $\sqrt{3}$ or $\sqrt[4]{6}$. 14. $\sqrt[4]{48}$ or $\sqrt[5]{64}$.

74. Multiplication of Surds. — It follows from the first principle in § 71 that

The product of two surds of the same order is equal to the surd of that order of which the number under the radical sign equals the product of the numbers under the radical signs in the given surds. That is,

$$\sqrt[n]{a}\sqrt[n]{b}=\sqrt[n]{ab}.$$

Example. — $\sqrt[3]{4}\sqrt[3]{5}=\sqrt[3]{20}$, and $\sqrt[4]{at^2}\sqrt[4]{t^3}=\sqrt[4]{at^5}=t\sqrt[4]{at}$.

The product of two surds of different orders may be obtained by first changing them to surds of the same order as in § 73, then applying the above principle.

Example. — Find the product of $\sqrt{2n}$, $\sqrt[3]{3n^2}$, and $\sqrt[6]{n^4}$.

$$\sqrt{2n} = \sqrt[6]{8n^3};\ \sqrt[3]{3n^2} = \sqrt[6]{9n^4};\ \sqrt[6]{n^4} = \sqrt[6]{n^4}.$$

Hence,
$$\begin{aligned}\sqrt{2n}\,\sqrt[3]{3n^2}\,\sqrt[6]{n^4} &= \sqrt[6]{8n^3}\,\sqrt[6]{9n^4}\,\sqrt[6]{n^4}\\ &= \sqrt[6]{72n^{11}}, \text{ or } n\sqrt{72n^5}.\end{aligned}$$

To multiply two polynomials involving surds, proceed as in multiplication of polynomials in Chapter I.

Example. — Multiply $2\sqrt{5} + 5\sqrt{2}$ by $3\sqrt{5} - 4\sqrt{2}$.

$$\begin{array}{rrr} & 2\sqrt{5} + 5\sqrt{2} & \\ & 3\sqrt{5} - 4\sqrt{2} & \\ \hline & 6\sqrt{25} + 15\sqrt{10} & \\ & - 8\sqrt{10} & - 20\sqrt{4} \\ \hline 30 + & 7\sqrt{10} & - 40, \text{ or } 7\sqrt{10} - 10 \end{array}$$

A term consisting of a surd factor with a coefficient may be expressed as an *entire surd* by changing the coefficient to the form of a surd and then multiplying.

Example. — Write as an entire surd $3\sqrt[3]{4}$.

We may write $\quad 3 = \sqrt[3]{3^3} = \sqrt[3]{27}.$

Hence, $\quad 3\sqrt[3]{4} = \sqrt{27}\,\sqrt[3]{4} = \sqrt[3]{108}.$

EXERCISES

Find the product of:

1. $\sqrt{5}\,\sqrt{7}$.
2. $\sqrt{2}\,\sqrt{3}\,\sqrt{6}$.
3. $\sqrt{8}\,\sqrt{2}\,\sqrt{8}$.
4. $\sqrt[3]{9}\,\sqrt[3]{6}$.
5. $\sqrt[3]{21}\,\sqrt[3]{49}$.
6. $\sqrt[4]{2}\,\sqrt[4]{8}\,\sqrt[4]{5}$.

7. $\sqrt[5]{9}\ \sqrt[5]{40}$.

8. $\sqrt{\frac{1}{2}}\ \sqrt{\frac{1}{8}}$.

9. $\sqrt{\frac{4}{5}}\ \sqrt{\frac{5}{8}}$.

10. $\sqrt{\frac{2}{3}}\ \sqrt{\frac{3}{5}}\ \sqrt{\frac{5}{2}}$.

11. $\sqrt{\frac{1}{2}}\ \sqrt{\frac{1}{4}}\ \sqrt{\frac{1}{8}}$.

12. $\sqrt{2}\ \sqrt[3]{3}$.

13. $\sqrt[4]{x^3}\ \sqrt[5]{x^2}$.

14. $\sqrt{2n}\ \sqrt[3]{5n^2}$.

15. $\sqrt[4]{3t^2}\ \sqrt{2t}\ \sqrt[3]{t}$.

16. $\sqrt{2}\ \sqrt[3]{3}\ \sqrt[4]{4}$.

17. $\sqrt[6]{5y^5}\sqrt{2y}\ \sqrt[3]{3y^2}$.

18. $\sqrt[4]{2V^3}\sqrt[3]{3V}\sqrt{V}$.

19. $\sqrt{a}\ \sqrt{a-b}$.

20. $\sqrt{m-n}\ \sqrt{m+n}$.

21. $\sqrt{x+y}\ \sqrt{x^2-y^2}$.

22. $(\sqrt{3}+\sqrt{2})(\sqrt{3}-\sqrt{2})$.

23. $(5\sqrt{2}+2\sqrt{3})(\sqrt{2}+3\sqrt{3})$.

24. $(\sqrt{8}+\sqrt{5})(\sqrt{2}-\sqrt{5})$.

25. $(1+\sqrt{2}+\sqrt{3})(1-\sqrt{2}+\sqrt{3})$.

26. $(\sqrt{2}+\sqrt{3}+\sqrt{5})(\sqrt{2}+\sqrt{3}-\sqrt{5})$.

27. $(\sqrt{a}+\sqrt{b})(\sqrt{a}+2\sqrt{b})$.

28. $(m\sqrt{n}-\sqrt{mn}+n\sqrt{m})(\sqrt{m}-\sqrt{n})$.

29. $(\sqrt{x}-\sqrt[4]{xy}+\sqrt{y})(\sqrt{x}+\sqrt[4]{xy}+\sqrt{y})$.

30. $(\sqrt{n-1}+3)(\sqrt{n-1}+5)$.

31. $(\sqrt[3]{a^2}-\sqrt[3]{ab}+\sqrt[3]{b^2})(\sqrt[3]{a}+\sqrt[3]{b})$.

32. $(\sqrt{p}+\sqrt{q}+\sqrt{r})^2$.

Simplify:

33. $(\frac{1}{2}R\sqrt{2+\sqrt{2}})(\frac{1}{2}R\sqrt{2-\sqrt{2}})$.

34. $\sqrt{R^2-(\frac{1}{2}R\sqrt{2-\sqrt{2}})^2}$.

Express as entire surds:

35. $2\sqrt{3}$.

36. $5\sqrt{7}$.

37. $\frac{1}{2}\sqrt{2}$.

38. $2\sqrt[3]{2}$.

39. $\frac{1}{8}\sqrt[3]{3}$.

40. $a\sqrt{ab}$.

41. $t^3\sqrt[4]{2t}$.

42. $4mn\sqrt[3]{n^2}$.

43. $\frac{a}{b}\sqrt{\frac{b}{a}}$.

44. $\frac{3v}{2t^2}\sqrt[3]{\frac{16t}{27v^2}}$.

45. $\frac{2c}{k}\sqrt[4]{\frac{k^3}{16c^2}}$.

75. Division of Surds. — The division of two surds, or of two expressions involving surds, is most easily performed by first indicating the division in the form of a fraction, and then multiplying both dividend and divisor by such an expression as will free the divisor of surds.

This process is called **rationalizing the divisor.** The multiplier used is called the **rationalizing factor.**

When the divisor is a monomial, the rationalizing factor is a monomial.

EXAMPLE. — Divide $24\sqrt{2}$ by $\sqrt{12}$.

Multiplying both dividend and divisor by $\sqrt{3}$,

$$\frac{24\sqrt{2}}{\sqrt{12}} = \frac{24\sqrt{6}}{\sqrt{36}} = \frac{24\sqrt{6}}{6} = 4\sqrt{6}.$$

When the divisor is a binomial of the form $\sqrt{a} + \sqrt{b}$, the rationalizing factor is $\sqrt{a} - \sqrt{b}$. And when the divisor is of the form $\sqrt{a} - \sqrt{b}$, the rationalizing factor is $\sqrt{a} + \sqrt{b}$. The product of $\sqrt{a} + \sqrt{b}$ and $\sqrt{a} - \sqrt{b}$ is $a - b$.

Two expressions of the forms $\sqrt{a} + \sqrt{b}$ and $\sqrt{a} - \sqrt{b}$ are called **conjugate expressions.**

EXAMPLE. — Divide $2\sqrt{5} + 3\sqrt{2}$ by $\sqrt{5} + \sqrt{2}$.

Multiplying by $\sqrt{5} - \sqrt{2}$, conjugate of $\sqrt{5} + \sqrt{2}$,

$$\frac{2\sqrt{5} + 3\sqrt{2}}{\sqrt{5} + \sqrt{2}} = \frac{(2\sqrt{5} + 3\sqrt{2})(\sqrt{5} - \sqrt{2})}{(\sqrt{5} + \sqrt{2})(\sqrt{5} - \sqrt{2})} = \frac{4 + \sqrt{10}}{3} = \frac{4}{3} + \frac{1}{3}\sqrt{10}.$$

When the divisor is a trinomial of the form $\sqrt{a} \pm \sqrt{b} \pm \sqrt{c}$, although it may be rationalized by a single multiplication, it is easier to perform two multiplications in succession. By multiplying by the trinomial obtained by changing the sign of one term of the divisor, a new divisor is obtained that contains one less surd than before.

Example. — Divide 2 by $\sqrt{5}+\sqrt{2}-\sqrt{3}$.

Multiplying by $\sqrt{5}+\sqrt{2}+\sqrt{3}$,

$$\frac{2}{\sqrt{5}+\sqrt{2}-\sqrt{3}}=\frac{2(\sqrt{5}+\sqrt{2}+\sqrt{3})}{(\sqrt{5}+\sqrt{2}-\sqrt{3})(\sqrt{5}+\sqrt{2}+\sqrt{3})}$$

$$=\frac{2(\sqrt{5}+\sqrt{2}+\sqrt{3})}{4+2\sqrt{10}},\text{ or }\frac{\sqrt{5}+\sqrt{2}+\sqrt{3}}{2+\sqrt{10}}.$$

Now multiplying by $2-\sqrt{10}$,

$$\frac{\sqrt{5}+\sqrt{2}+\sqrt{3}}{2+\sqrt{10}}=\frac{-3\sqrt{2}+2\sqrt{3}-\sqrt{30}}{-6}=\tfrac{1}{2}\sqrt{2}-\tfrac{1}{3}\sqrt{3}+\tfrac{1}{6}\sqrt{30}.$$

Evidently, a similar process may be used when the divisor is a polynomial of more than three terms, if the surds are of the second order. More complicated problems are not given here.

EXERCISES

Perform the indicated divisions:

1. $\dfrac{2}{\sqrt{3}}$.
2. $\dfrac{\sqrt{2}}{\sqrt{5}}$.
3. $\dfrac{8}{\sqrt{6}}$.
4. $\dfrac{12}{\sqrt{18}}$.
5. $\dfrac{9}{\sqrt{21}}$.
6. $\dfrac{\sqrt{8}}{\sqrt{10}}$.
7. $\dfrac{\sqrt{12}}{\sqrt{24}}$.
8. $\dfrac{5}{\sqrt[3]{2}}$.
9. $\dfrac{6}{\sqrt[3]{3}}$.
10. $\dfrac{\sqrt[3]{9}}{\sqrt[3]{18}}$.
11. $\dfrac{\sqrt{m}}{\sqrt{n}}$.
12. $\dfrac{10r}{\sqrt{2r}}$.
13. $\dfrac{15V}{\sqrt{5V}}$.
14. $\dfrac{\sqrt[3]{4y}}{\sqrt[3]{2y^2}}$.
15. $\dfrac{\sqrt{x}}{\sqrt[3]{x}}$.
16. $\dfrac{36}{\sqrt{3}+1}$.
17. $\dfrac{10}{1-\sqrt{5}}$.
18. $\dfrac{\sqrt{2}}{1+\sqrt{2}}$.
19. $\dfrac{\sqrt{3}+1}{\sqrt{3}-1}$.
20. $\dfrac{\sqrt{2}+\sqrt{5}}{2\sqrt{2}+\sqrt{3}}$.
21. $\dfrac{6\sqrt{7}-5\sqrt{6}}{2\sqrt{7}+3\sqrt{6}}$.
22. $\dfrac{1}{\sqrt{m}-\sqrt{n}}$.
23. $\dfrac{a-b}{\sqrt{a}+\sqrt{b}}$.
24. $\dfrac{x^3-y^3}{\sqrt{x^3}-\sqrt{y^3}}$.

25. $\dfrac{1-\sqrt{2}}{1+\sqrt{2}+\sqrt{3}}$.

26. $\dfrac{5}{\sqrt{7}-\sqrt{5}+\sqrt{3}}$.

27. $\dfrac{\sqrt{2}-\sqrt{3}+\sqrt{5}}{\sqrt{2}+\sqrt{3}+\sqrt{5}}$.

28. $\dfrac{\sqrt{p}}{\sqrt{p}+\sqrt{q}+\sqrt{r}}$.

29. The volume of a frustum of a pyramid whose altitude is H, lower base B, and upper base b is

$$\frac{\frac{1}{3}H(B\sqrt{B}-b\sqrt{b})}{\sqrt{B}-\sqrt{b}}.$$

Show that this may be written in the simpler form

$$\tfrac{1}{3}H(B+b+\sqrt{Bb}).$$

76. Powers of Surds. — Powers of surds are obtained by use of the principle

$$(\sqrt[m]{a})^n = \sqrt[m]{a^n}.$$

This principle follows from § 74, thus:

$$\begin{aligned}(\sqrt[m]{a})^n &= \sqrt[m]{a}\,\sqrt[m]{a}\,\sqrt[m]{a}\cdots \text{to } n \text{ factors}\\ &= \sqrt[m]{a\cdot a\cdot a\cdots \text{to } n \text{ factors}}\\ &= \sqrt[m]{a^n}.\end{aligned}$$

Example. — Raise $2a\sqrt[3]{2a^2b}$ to the fourth power.

$$(2a\sqrt[3]{2a^2b})^4 = 16a^4\sqrt[3]{(2a^2b)^4} = 16a^4\sqrt[3]{16a^8b^4} = 32a^6b\sqrt[3]{2a^2b}.$$

77. Roots of Surds. — Roots of surds are obtained by use of the principle

$$\sqrt[m]{\sqrt[n]{a}} = \sqrt[mn]{a}.$$

This principle follows from the fact that the mth root of the nth root of a number is one of the m equal factors of one of the n

equal factors of the number, which is one of the mn equal factors of the number.

Example. — Find the fifth root of $\sqrt[4]{3x^3y^2}$.

$$\sqrt[5]{\sqrt[4]{3x^3y^2}} = \sqrt[20]{3x^3y^2}.$$

EXERCISES

Simplify:

1. $(\sqrt{3})^3$.
2. $(\sqrt{5})^4$.
3. $(2\sqrt{2})^3$.
4. $(4\sqrt[3]{2})^2$.
5. $(3\sqrt[3]{5})^4$.
6. $(\sqrt[3]{x^2})^2$.
7. $(\sqrt[4]{N^3})^3$.
8. $(t\sqrt[5]{t})^6$.
9. $(2ab\sqrt{ab})^3$.
10. $(6x^2\sqrt[4]{x^2})^2$.
11. $(10m^3\sqrt[5]{mn^3})^4$.
12. $(3p^3q^2\sqrt[5]{p^2q^4})^5$.
13. $(2\sqrt{a^2-b^2})^4$.
14. $\sqrt[3]{\sqrt{5}}$.
15. $\sqrt[3]{\sqrt{8}}$.
16. $\sqrt{\sqrt[3]{25}}$.
17. $\sqrt{\sqrt[3]{64}}$.
18. $\sqrt[5]{\sqrt[3]{32}}$.
19. $\sqrt{\sqrt[3]{\sqrt[4]{x^6}}}$.
20. $\sqrt[4]{\sqrt[3]{625n^8}}$.
21. $\sqrt[3]{\sqrt[4]{8x^6y^{12}}}$.
22. $\sqrt{49r^6s^4\sqrt[3]{8s^{12}}}$.
23. $\sqrt{\sqrt[3]{(a^2-b^2)^2}}$.
24. $\sqrt[m]{\sqrt[n]{\sqrt[p]{a^{mn}}}}$.

78. Imaginary Numbers. — Imaginary numbers have been defined (see § 63) as the indicated even roots of negative numbers. By means of more advanced mathematics, any imaginary number may be expressed in terms of imaginary numbers of the second order. Hence the discussion is limited to the latter. Imaginary numbers will be involved in the roots of equations in the next and succeeding chapters.

A binomial consisting of a real and an imaginary term is called a **complex expression**.

Thus, $2-\sqrt{-5}$ is a complex expression.

79. Typical Form of Imaginary Numbers. — By applying the same process as that of simplifying a surd by removing a factor from under the radical sign, any imaginary number may be reduced to the form $n\sqrt{-1}$.

Thus, $\sqrt{-4} = \sqrt{4}\,\sqrt{-1} = 2\sqrt{-1}$; $\sqrt{-10} = \sqrt{10}\,\sqrt{-1}$.

When an imaginary number is expressed in the form $n\sqrt{-1}$, it is said to be in the **typical form**.

The factor $\sqrt{-1}$ of any imaginary number expressed in the typical form is called the **imaginary unit**. It is represented usually by the more convenient symbol i.

80. Addition and Subtraction of Imaginary Numbers. — Imaginary numbers to be added or subtracted must first be reduced to the typical form, and then combined as in the addition and subtraction of surds.

EXAMPLE. — Simplify $\sqrt{-16} + \sqrt{-9} - \sqrt{-25}$.

$$\begin{aligned}\sqrt{-16} + \sqrt{-9} - \sqrt{-25} &= 4i + 3i - 5i \\ &= 2i, \text{ or } 2\sqrt{-1}.\end{aligned}$$

To add or subtract two complex expressions, combine their real terms and combine their imaginary terms.

EXERCISES

Reduce to the typical form:

1. $\sqrt{-36}$. 4. $\sqrt{-100}$. 7. $\sqrt{-12}$. 10. $\sqrt{-\frac{1}{9}}$.
2. $\sqrt{-64}$. 5. $\sqrt{-625}$. 8. $\sqrt{-20}$. 11. $\sqrt{-a^4b^2}$.
3. $\sqrt{-81}$. 6. $\sqrt{-6}$. 9. $\sqrt{-\frac{1}{4}}$. 12. $\sqrt{-16\,m^2n^6}$.
13. $\sqrt{-a^2-2\,ab-b^2}$. 14. $\sqrt{24\,xy-9\,x^2-16\,y^2}$.

Simplify:

15. $\sqrt{-9}-\sqrt{-4}+\sqrt{-36}$. 18. $\sqrt{-49}-2\sqrt{-1}+\sqrt{-121}$.
16. $\sqrt{-81}+\sqrt{-100}-\sqrt{-1}$. 19. $\sqrt{-\frac{2}{3}}+\sqrt{-\frac{3}{2}}+\sqrt{-1}$.
17. $\sqrt{-64}-\sqrt{-4}-\sqrt{-16}$. 20. $a\sqrt{-a^2}+\sqrt{-a^4}+a^2\sqrt{-1}$.

21. Add $x + 2y\sqrt{-1}$ and $2x - y\sqrt{-1}$.

22. Add $10 - 3i$ and $6 + 5i$.

23. From $3m - 2ni$ take $m + 3ni$.

24. Simplify $(8 - 5\sqrt{-9}) - (12 + 2\sqrt{-16})$.

25. Write in the form of a complex expression $(a - bi) - (c + di)$

81. Powers of i. — The successive powers of i are:

$$\begin{aligned} i &= \sqrt{-1} = i \\ i^2 &= (\sqrt{-1})^2 = -1 \\ i^3 &= (\sqrt{-1})^3 = i \times i^2 = -i \\ i^4 &= (\sqrt{-1})^4 = i \times i^3 = -i^2 = +1 \\ i^5 &= (\sqrt{-1})^5 = i \times i^4 = i \\ i^6 &= (\sqrt{-1})^6 = i \times i^5 = i^2 = -1 \end{aligned}$$

etc.

Hence, the successive powers of i have only the four different values, i, -1, $-i$, $+1$, repeating in regular order.

It follows that the value of any power of i may be determined by expressing it in terms of the four lowest powers.

Thus, $i^{11} = (i^8)(i^3) = (i^4)^2(i^3) = (+1)^2(-i) = -i$;

$i^{16} = (i^4)^4 = (+1)^4 = +1$.

82. Multiplication of Imaginary Numbers. — The principle $\sqrt[n]{a}\,\sqrt[n]{b} = \sqrt[n]{ab}$, used in the multiplication of surds, does not hold for the multiplication of imaginary numbers.

Thus, by that principle, $\sqrt{-4}\,\sqrt{-9} = \sqrt{36} = 6$.

But, $\sqrt{-4}\sqrt{-9} = 2i \times 3i = 6i^2 = -6$.

Hence, use of the principle gives the wrong value.

***Imaginary numbers to be multiplied should first be reduced to the typical form, and then the product formed by using the values of the powers of i in* § 81.**

Example 1. — Simplify $\sqrt{-16}\,\sqrt{-9}\,\sqrt{-25}$.

$$\begin{aligned}\sqrt{-16}\,\sqrt{-9}\,\sqrt{-25} &= (4\,i)(3\,i)(5\,i)\\ &= 60\,i^3\\ &= -60\,i, \text{ or } -60\sqrt{-1}.\end{aligned}$$

Example 2. — Multiply $\sqrt{-16}-1$ by $\sqrt{-9}+2$.

$$\sqrt{-16}-1 = 4\,i-1\,;\quad \sqrt{-9}+2 = 3\,i+2.$$

$$\begin{array}{r} 4\,i-1\\ \underline{3\,i+2}\\ 12\,i^2-3\,i\\ \underline{+8\,i-2}\\ -12+5\,i-2, \text{ or } 5\sqrt{-1}-14.\end{array}$$

EXERCISES

Find the value of:

1. i^9.
2. i^{12}.
3. i^{19}.
4. i^{21}.
5. i^{36}.
6. i^{42}.
7. i^{51}.
8. i^{57}.

Simplify:

9. $\sqrt{-25}\,\sqrt{-4}$.
10. $\sqrt{-9}\,\sqrt{-16}$.
11. $\sqrt{-36}\,\sqrt{-49}$.
12. $\sqrt{-81}\,\sqrt{-64}$.
13. $\sqrt{-100}\,\sqrt{-1}$.
14. $\sqrt{-a^2}\,\sqrt{-b^2}$.
15. $\sqrt{-10}\,\sqrt{-6}$.
16. $\sqrt{+4}\,\sqrt{-9}\,\sqrt{-25}$.
17. $\sqrt{-16}\,\sqrt{-49}\,\sqrt{-\frac{1}{4}}$.
18. $\sqrt{-\frac{1}{9}}\,\sqrt{-81}\,\sqrt{-36}$.
19. $\sqrt{-5}\,\sqrt{-7}\,\sqrt{-2}$.
20. $\sqrt{-n}\,\sqrt{-n^3}\,\sqrt{-n^5}$.
21. $\sqrt{-25}\,\sqrt{-4}\,\sqrt{-36}\,\sqrt{-49}\,\sqrt{-1}$.
22. $\sqrt{-16}\,\sqrt{-9}\,\sqrt{-25}\,\sqrt{-81}\,\sqrt{-2}\,\sqrt{-2}$.
23. $(1+\sqrt{-2})(1-\sqrt{-2})$.
24. $(\sqrt{-4}+3)(\sqrt{-4}-3)$.

25. $(\sqrt{-9}-\sqrt{-16})(\sqrt{-4}+\sqrt{-1})$.

26. $(\sqrt{-2}+\sqrt{-3})(\sqrt{-2}-\sqrt{-3})$.

27. $(2\sqrt{-4}+3\sqrt{-9})(2\sqrt{-4}+3\sqrt{-9})$.

28. $(\sqrt{-a}+\sqrt{-b})(\sqrt{-a}-\sqrt{-b})$.

29. $(x\sqrt{-x}-y\sqrt{-y})(x\sqrt{-x}-y\sqrt{-y})$.

30. Show that the product of two complex expressions of the forms $a+bi$ and $a-bi$ is real.

83. Division of Imaginary Numbers. — In division involving imaginary numbers it is best to first reduce each imaginary number to the typical form, then multiply both dividend and divisor by such an expression as will make the divisor real and free of radicals. The process is similar to that in the division of surds.

Example 1. — Divide 6 by $\sqrt{-4}$.

$$\frac{6}{\sqrt{-4}}=\frac{6}{2i}=\frac{6i}{2i^2}=\frac{6i}{-2}=-3i, \text{ or } -3\sqrt{-1}.$$

Example 2. — Divide $2-\sqrt{-9}$ by $3+\sqrt{-16}$.

$$\frac{2-\sqrt{-9}}{3+\sqrt{-16}}=\frac{2-3i}{3+4i}=\frac{(2-3i)(3-4i)}{(3+4i)(3-4i)}=\frac{-6-17i}{25}=-\frac{6}{25}-\frac{17}{25}i.$$

EXERCISES

Simplify:

1. $\dfrac{5}{\sqrt{-1}}$.

2. $\dfrac{3}{2\sqrt{-1}}$.

3. $\dfrac{10}{\sqrt{-4}}$.

4. $\dfrac{21}{\sqrt{-9}}$.

5. $\dfrac{n}{\sqrt{-n}}$.

6. $\dfrac{4\sqrt{-1}}{\sqrt{-4}}$.

7. $\dfrac{\sqrt{-16}}{\sqrt{-9}}$.

8. $\dfrac{\sqrt{-100}}{\sqrt{-25}}$.

9. $\dfrac{-t}{\sqrt{-9t^2}}$.

10. $\dfrac{2\sqrt{-25a^4}}{3\sqrt{-16a^2}}$.

11. $\dfrac{\sqrt{-25v^8}}{\sqrt{-49v^6}}$.

12. $\dfrac{\sqrt{-\frac{1}{4}x^2}}{\sqrt{-\frac{1}{9}x^4}}$.

13. $\dfrac{2}{1+\sqrt{-9}}$.

14. $\dfrac{1+\sqrt{-3}}{1-\sqrt{-2}}$.

15. $\dfrac{3-\sqrt{-5}}{3+\sqrt{-5}}$.

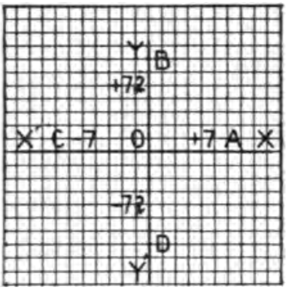
16. $\dfrac{2+\sqrt{-6}}{1-\sqrt{-8}}$.

17. $\dfrac{1}{5+2\sqrt{-5}}$. 18. $\dfrac{\sqrt{-2}+2\sqrt{-3}}{2\sqrt{-5}-3\sqrt{-2}}$. 19. $\dfrac{x+\sqrt{-y}}{x-\sqrt{-y}}$.

20. Show that the quotient of two complex expressions of the forms $a+bi$ and $a-bi$ is a complex expression.

84. Graphical Representation of Imaginary Numbers. — Imaginary numbers were so named centuries ago because when first discovered they had no meaning, and were for centuries uninterpreted. It is now seen that they are truly numbers, and that they may be represented graphically.

All positive and negative real numbers may be represented, as we have seen before, by means of distances measured to the right or left of a starting point O on a line XX'.



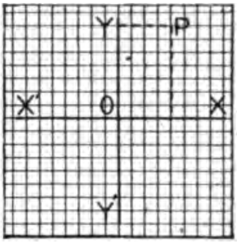
If the line OA, which represents $+7$, is revolved counterclockwise about O, it may be made to take the position of OB, OC, OD, or any other line through O.

Now, multiplying $+7$ by i gives $+7i$. Multiplying again by i gives -7, which is represented by OC. Hence multiplying $+7$ twice by i gives the same result as revolving OA into the position OC. Therefore, multiplying $+7$ once by i may be interpreted as revolving OA only half as far, *i.e.* into the position OB. Hence, $+7i$ is represented graphically by OB. Similarly, multiplying $+7$ three times by i would revolve it into the position OD. Hence, $-7i$ is represented graphically by OD. The same interpretation would hold in case of any other real number that might be chosen in place of $+7$. Hence, the following principle:

If all real numbers are represented by distances measured along the line XX' from O, positive numbers to the right and negative numbers to the left, then all imaginary numbers will be represented by distances measured from O along the line YY' perpendicular to XX', those with positive coefficients above O, and those with negative coefficients below.

Example. — $\sqrt{-9}=3i$. Hence $\sqrt{-9}$ is represented by a distance of 3 spaces measured upward on YY' from O.

85. Graphical Representation of Complex Expressions. — Since a complex expression is composed of a real term and an imaginary term, it may be represented graphically.



Thus, to represent $5 + \sqrt{-49}$, or $5 + 7i$, we measure the 5 real units along OX and the 7 imaginary units along OY. As in other graphical work, this locates the point P. The point P represents graphically the complex expression $5 + \sqrt{-49}$.

EXERCISES

Represent graphically:

1. $\sqrt{-4}$.
2. $\sqrt{-16}$.
3. $\sqrt{-36}$.
4. $-\sqrt{-25}$.
5. $-\sqrt{-81}$.
6. $-\sqrt{-64}$.
7. $8 + \sqrt{-9}$.
8. $12 - \sqrt{-16}$.
9. $7 + \sqrt{-100}$.
10. $-6 + \sqrt{-25}$.
11. $-4 - \sqrt{-81}$.
12. $-12 - \sqrt{-36}$.

SUPPLEMENTARY EXERCISES

Find to three decimal places the square root of:

1. $\frac{2}{5}$.

SUGGESTION. — Either change a common fraction to a decimal, or rationalize the denominator, before extracting the square root.

2. $\frac{3}{7}$. 3. $\frac{5}{11}$. 4. $\frac{7}{19}$. 5. $\frac{5}{6}$. 6. $\frac{3}{10}$. 7. $\frac{13}{20}$.

Simplify:

8. $\sqrt[n]{a^{2n}b^{3n}}$.
9. $\sqrt[2n]{x^{6n}y^{n}}$.
10. $\sqrt[n+1]{A^{2n+2}B^{n+1}}$.
11. $x\sqrt[2n]{x^{3n}}$.
12. $\sqrt[n]{R^{n} - 3R^{n}\sqrt{5}}$.
13. $(\sqrt[n]{a} - \sqrt[n]{b})^2$.
14. Does $-3 + \sqrt{11}$ satisfy $x^2 + 6x - 2 = 0$?
15. Does $4 - \sqrt{17}$ satisfy $n^2 - 8n - 1 = 0$?
16. Is $-\frac{1}{2} + \frac{1}{2}\sqrt{-3}$ a cube root of 1?

17. Is $-\frac{1}{2}-\frac{1}{2}\sqrt{-3}$ a cube root of 1?

18. Find the square root of $7+2\sqrt{10}$.

SOLUTION. — Since $(\sqrt{a}\pm\sqrt{b})^2=a\pm 2\sqrt{ab}+b$, the square root of any expression of the form $x\pm 2\sqrt{y}$ may be found, if it can be written in the form $a\pm 2\sqrt{ab}+b$, by taking the sum, or difference, of the square roots of the terms a and b.

Now, $7+2\sqrt{10}$ may be written $5+2\sqrt{5\times 2}+2$. Hence its square root is $\sqrt{5}+\sqrt{2}$.

Find the square root of:

19. $10-\sqrt{84}$.

SUGGESTION. — This may be written $10-2\sqrt{21}$, or $7-2\sqrt{7\times 3}+3$.

20. $11+2\sqrt{30}$.

21. $14+2\sqrt{45}$.

22. $13-\sqrt{48}$.

23. $6+\sqrt{32}$.

24. $26-\sqrt{640}$.

25. $18+\sqrt{308}$.

Rationalize the denominators of:

26. $\dfrac{3}{2-\sqrt{-3}}$.

27. $\dfrac{2}{\sqrt{3}+\sqrt{-1}}$.

28. $\dfrac{2-\sqrt{-3}}{2+\sqrt{-3}}$.

29. $\dfrac{3}{\sqrt{-3}-\sqrt{2}}$.

30. $\dfrac{x+y\sqrt{-1}}{x-y\sqrt{-1}}$.

31. $\dfrac{a+b\sqrt{-1}}{a\sqrt{-1}-b}$.

Simplify:

32. $(-\frac{1}{2}-\frac{1}{2}\sqrt{-5})(-\frac{1}{2}-\frac{1}{2}\sqrt{-5})^2$.

33. $(2\sqrt{3}+3\sqrt{-7})(5\sqrt{3}-4\sqrt{-7})$.

CHAPTER VII

QUADRATIC EQUATIONS

86. Quadratic Equations. — Every **quadratic equation** (see § 42 for definition) in one unknown number whose unknown number is denoted by x can be written in the form $Ax^2 + Bx + C = 0$, by simplifying, transposing, and combining similar terms.

Thus, $(5x - 2)(2x + 1) = 7x^2 - 4x$ may be written $3x^2 + 5x - 2 = 0$.

If a quadratic equation, when reduced to the above form, contains a term involving the first power of the unknown number, it is called a **complete quadratic** equation. One that does not contain a term involving the first power of the unknown number is called a **pure quadratic equation.**

Thus, $5x^2 - 7x + 2 = 0$ is a *complete* quadratic equation. And $2N^2 + 9 = 0$ is a *pure* quadratic equation.

EXERCISES

Express each of the following equations in the form $Ax^2 + Bx + C = 0$, and determine whether it is a complete or pure quadratic:

1. $x(2x - 3) = 10 - 3x$.
2. $5n(n + 2) = n^2 + 2$.
3. $12t^2 = 7(t - 3)$.
4. $10(V - 1) = 3V^2 + 2$.
5. $(2a - 1)^2 = 3a - 5$.
6. $(y + 2)^3 = (y - 1)^3$.
7. $(6r - 5)(r + 2) = (2r + 7)(2r - 1)$.
8. $(2P + 1)(2P - 1) = (3P - 2)(3P + 2)$.
9. $(a - 1)(a - 2)(a - 3) - (a + 1)(a + 2)(a + 3) = 0$.
10. $\dfrac{r^2 + r + 1}{r - 1} - 6 = \dfrac{r^2 - r + 1}{r + 1}$.
11. $2T + 1 + \dfrac{(3T + 1)(T - 2)}{T + 2} = 0$.

87. Quadratics solved by Factoring. — Some quadratic equations may be solved by factoring, as illustrated in the following example.

EXAMPLE. — Solve $2t^2 + t = 10$.

Transposing, $2t^2 + t - 10 = 0$.

Factoring, $(2t + 5)(t - 2) = 0$.

Now, this equation is satisfied by any value of t that will make either of the factors $2t + 5$ or $t - 2$ zero, since then their product would be zero. Such values of t may be found by setting each factor equal to zero, and solving the resulting linear equations. Setting each factor equal to zero,

$$2t + 5 = 0;\ \text{whence } t = -2\tfrac{1}{2}.$$

$$t - 2 = 0;\ \text{whence } t = 2.$$

To solve a quadratic equation by factoring:

(1) *Write the equation in the form* $Ax^2 + Bx + C = 0$.

(2) *Factor the resulting first member.*

(3) *Set each of the factors equal to zero, and solve the resulting linear equations.*

Since a quadratic equation that can be solved by factoring is *equivalent* to the two linear equations to which it leads, and each of these has always one root, the quadratic equation has two roots. It can be shown that in general,

Every quadratic equation has two roots.

EXERCISES

Solve by factoring:

1. $n^2 - 5n + 4 = 0$.
2. $k^2 = 12k - 32$.
3. $P^2 + 6P = 16$.
4. $2A^2 = A + 3$.
5. $5v + 6 = 4v^2$.
6. $18t^2 + 61t = 7$.
7. $10k^2 = 31k + 14$.
8. $81N + 24 = 60N^2$.
9. $x(x + 2) = 4x + 15$.
10. $5(p + 25) = 6p^2$.
11. $(4d - 1)^2 = 10d^2 + 31$.
12. $(x + 6)^2 = (3x - 2)^2$.

13. $(n-1)(n+1)(n-2)=n^2(n+2)-103.$

14. $(2R-3)(R+4)=(3R-14)(2R+3)+R^2-6.$

15. $6-\frac{4}{y^2}=\frac{23}{y}.$

16. $\frac{h+3}{h+15}=\frac{1}{h+8}.$

17. $2D+1=\frac{3D^2-4}{D+2}.$

18. $\frac{1}{2a-2}=\frac{a-7}{a+1}+\frac{24+a}{5-5a^2}.$

19. $\frac{x^2+x+1}{x+1}=\frac{x^2-x+1}{x-1}-\frac{2}{3}.$

20. $\frac{5}{g+5}-\frac{2}{g+2}=\frac{6}{g+23}.$

88. Quadratics solved by Square Root.—Any pure quadratic equation may be solved by square root.

EXAMPLE 1.—Solve $4n^2-1=0$.

Transposing, $4n^2=1.$

Dividing by 4, $n^2=\frac{1}{4}.$

Now since $n^2=\frac{1}{4}$, n must be the square root of $\frac{1}{4}$.

Hence, $n=\pm\frac{1}{2}.$

EXAMPLE 2.—Solve $2V^2+3=0$.

Transposing, $2V^2=-3.$

Dividing by 2, $V^2=-\frac{3}{2}.$

Taking square root, $V=\pm\sqrt{-\frac{3}{2}}$, or $\pm\frac{1}{2}\sqrt{-6}$, *imaginary* roots.

To solve a pure quadratic equation by square root:

(1) *By transposing, etc., solve for the square of the unknown number.*

(2) *Find the square roots of the value obtained.*

EXERCISES

Solve:

1. $5y^2-20=0.$

2. $12F^2-125=7F^2.$

3 $k^2=63-6k^2.$

4. $16A^2=9.$

5. $3p^2-7=0.$

6. $8T^2+1=0.$

7. $5n^2+6=0.$

8. $12G^2+3=0.$

9. $7b^2-21=0.$

10. $16+5R^2=0.$

11. $x(x-2)=9-2x.$

12. $(N+3)(2N+5)=11(N+4).$

13. $\dfrac{1}{a+2}=\dfrac{a-2}{12}.$

14. $\dfrac{z^2+7}{9}=3.$

15. $\dfrac{A+1}{A-1}+\dfrac{A-1}{A+1}=1\tfrac{5}{6}.$

16. $\dfrac{6n^2-5}{n^2-4}-\dfrac{n-2}{n+2}=\dfrac{n+2}{n-2}.$

17. $\dfrac{t+9}{9t+1}-\dfrac{9t+1}{t+9}=0.$

18. $3a+2-\dfrac{(a+4)(4a-3)}{2a+3}=0.$

19. $\dfrac{K+7}{5}-\dfrac{9}{K-2}=1.$

20. $\dfrac{4m+3}{4m-3}+\dfrac{4m-3}{4m+3}=2\tfrac{4}{63}.$

21. $\dfrac{12}{5}\left(B-1+\dfrac{2}{B+1}\right)=\dfrac{5}{B+1}.$

Find approximate to hundredths the roots of:

22. $x^2=6.$

23. $4M^2-7=0.$

24. $15t^2-9=0.$

25. $1-5t^2=0.$

26. $3D^2=28.$

27. $8s^2-16=0.$

28. $\dfrac{H+5}{6}-\dfrac{7}{H+1}=1.$

89. Complete Quadratics solved by Completing the Square. — A trinomial that is a perfect square of a binomial can be written in either the form $a^2+2ab+b^2$ or $a^2-2ab+b^2$. (See § 15.) Hence, if we have given either a^2+2ab or a^2-2ab, it is made a perfect square by adding b^2, *i.e. the square of one half of the coefficient of a in the term 2 ab.*

This process is called **completing the square**.

Thus, to complete the square of k^2-12k, we add the square of $\frac{1}{2}$ of 12, or 36. This gives $k^2-12k+36$, which is the square of $k-6$.

By use of this process of completing the square, any complete quadratic equation can be solved.

EXAMPLE. — Solve $4y^2=6y-3$.

Transposing, $4y^2-6y=-3.$

Dividing by 4, $y^2-\frac{3}{2}y=-\frac{3}{4}.$

Adding square of $\frac{1}{2}$ of $-\frac{3}{2}$, $y^2-\frac{3}{2}y+\frac{9}{16}=-\frac{3}{16}.$

Taking square root, $y - \frac{1}{2} = \pm \frac{1}{2}\sqrt{-3}$.

Transposing, $y = \frac{1}{2} \pm \frac{1}{2}\sqrt{-3}$.

To solve a complete quadratic by completing the square:

(1) *Reduce the equation to the form* $x^2 + px = q$, *where* x *is the unknown number.*

(2) *Add to each member the square of one half the coefficient of the term in the first power of the unknown number.*

(3) *Take the square root, attaching the double sign to the second member.*

(4) *Solve the resulting linear equations.*

EXERCISES

Solve by completing the square:

1. $n^2 + 10n = 24$.
2. $4A^2 + 11A = 3$.
3. $3x^2 = 20 - 7x$.
4. $6v^2 = 49v - 8$.
5. $12T^2 + 31T - 15 = 0$.
6. $3y^2 + 8y + 4 = 0$.
7. $6N^2 - 2 = N$.
8. $7k = 10k^2 - 12$.
9. $7D^2 = 12D + 3$.
10. $4s^2 - 3 = 11s$.
11. $10z^2 - 29z + 10 = 0$.
12. $12H^2 - 14H + 3 = 0$.
13. $6t^2 + 6 = 13t$.
14. $2R^2 + R = 1$.
15. $3V^2 = 17V + 28$.
16. $9m^2 - 26 = 6m$.
17. $5 + G = 6G^2$.
18. $2E - 7 = 3E^2$.
19. $(3x - 4)^2 - (x - 2)^2 = 1$.
20. $\frac{1}{n} + \frac{1}{2 + n} - \frac{3}{4} = 0$.
21. $\frac{3v - 1}{v + 2} - \frac{4v - 1}{v + 3} = 1$.
22. $\frac{a + 5}{2a - 1} + \frac{3}{a - 4} = \frac{1}{3}$.
23. $\frac{1}{m + 2} - \frac{2m - 1}{4m + 3} + \frac{4}{5} = 0$.
24. $\frac{4}{d - 4} - 2 = \frac{d + 3}{d^2 - 5d + 4}$.
25. $\frac{y + 5}{3 - y} + \frac{2y}{2 - y} = \frac{8 - 3y}{y}$.
26. $\frac{5}{N - 2} + \frac{2}{N - 4} = \frac{3}{6 - N}$.
27. $\frac{2}{2t - 6} + \frac{8}{8 - 3t} + \frac{7}{t + 3} = 0$.
28. $\frac{x^2 - 7x + 2}{x^2 + 9x + 8} = \frac{x + 12}{x - 1}$.

Find approximate to hundredths the roots of:

29. $a^2+8a+4=0$.

30. $V^2+5V+1=0$.

31. $2n^2-n-2=0$.

32. $6D^2+3D=5$.

90. Quadratics solved by Formula. — Any quadratic equation whose unknown number is x may be written in the form

$$Ax^2+Bx+C=0.$$

If this general equation is solved by completing the square, the roots obtained are

$$x=\frac{-B\pm\sqrt{B^2-4AC}}{2A}.$$

Now, the roots of any given particular quadratic equation may be written down at once, without performing all of the steps in the solution by completing the square, by substituting for A, B, and C, in the above formula for the roots of the general equation Ax^2+Bx+C, the particular values which they have in the given equation.

EXAMPLE. — Solve $2x^2=1-5x$.

Transposing, $2x^2+5x-1=0$.

Now, comparing this equation to $Ax^2+Bx+C=0$, it is seen that $A=2$, $B=5$, and $C=-1$. Substituting these values for A, B, and C in the formula for the roots, we get

$$x=\frac{-5\pm\sqrt{25+8}}{4}$$

$$=-\tfrac{5}{4}\pm\tfrac{1}{4}\sqrt{33}.$$

The student should master this formula as thoroughly as the multiplication table, because by it the roots of complete quadratics can be obtained much more quickly than by completing the square.

EXERCISES

Solve by formula:

1. $Q^2-8Q+15=0.$
2. $z^2-2z-63=0.$
3. $a^2+16a+48=0.$
4. $2x^2-3x-5=0.$
5. $6v^2+37v+6=0.$
6. $6N^2+17N=14.$
7. $15b^2=8-2b.$
8. $21y^2=43y-20.$
9. $t^2-7t+2=0.$
10. $8B^2+B=5.$
11. $12n-7=2n^2.$
12. $20V^2=V-13.$
13. $3P^2-8=2P.$
14. $11k+3k^2-4=0.$
15. $9R^2-R-6=0.$
16. $(x-5)(x+2)=6.$
17. $x(x+8)+8(x-8)=0.$
18. $(p-4)^2=(3-p)(1+2p).$
19. $n^2+n+1=0.$
20. $n^2-n+1=0.$
21. $v^2=2v.$
22. $\dfrac{y-1}{y+1}=\dfrac{y+1}{y-1}-1.$
23. $\dfrac{1}{g}+g=1\tfrac{1}{2}.$
24. $R-\dfrac{1}{R}=3-\dfrac{1}{3}.$
25. $\dfrac{F-1}{F-4}+\dfrac{F+9}{F+3}=0.$
26. $b+2=\dfrac{3b^2-4}{2b+1}.$
27. $H=\dfrac{1}{H}-1.$
28. $\dfrac{n-2}{2+n}+\dfrac{2+n}{n-2}=\dfrac{2(5n-4)}{4-n^2}.$
29. $\dfrac{2}{5(c-2)}=\dfrac{1-c}{2+c}+\dfrac{c+1}{c^2-4}.$
30. $\dfrac{2r+3}{4r-2}-\dfrac{7-r}{2+2r}=\dfrac{7-3r}{4-3r}.$
31. $\dfrac{8+V}{8-V}+\dfrac{4-2V}{4+2V}=\dfrac{8}{3}.$
32. $\dfrac{1}{8}+\dfrac{1}{4(n+1)}=\dfrac{3}{2n^2-2}.$
33. $\dfrac{A+3}{2A+8}-\dfrac{A}{1+A}=\dfrac{1}{18}.$
34. $5Q-\dfrac{3Q-3}{Q-3}=2Q+\dfrac{3Q-6}{2}.$
35. $\dfrac{s-5}{s+3}-\dfrac{80}{s^2-9}=\dfrac{1}{2}+\dfrac{s-8}{3-s}.$

Find approximate to hundredths the roots of:

36. $f^2 + 6f + 5 = 0$.
37. $2S^2 - S - 2 = 0$.
38. $5v^2 = 2v + 2$.
39. $6n^2 + 5n = 2$.
40. $-x^2 + 4x + 6 = 0$.
41. $-2T^2 - T + 4 = 0$.

91. Literal Equations: Formulæ. — If a literal equation, or formula, is of the second degree in the number solved for, it may be solved by one of the foregoing methods.

EXAMPLE. — Solve $V^2 + 2aV = b^2 + 2ab$ for V.

Transposing and grouping, $V^2 + 2aV - (b^2 + 2ab) = 0$.

Using the formula, $A = 1$, $B = 2a$, and $C = -(b^2 + 2ab)$.

Hence,
$$V = \frac{-2a \pm \sqrt{4a^2 + 4(b^2 + 2ab)}}{2}$$
$$= \frac{-2a \pm (2a + 2b)}{2}$$
$$= b, \text{ or } -2a - b.$$

EXERCISES

1. Solve $av^2 - a^3 = 0$ for v.
2. Solve $x^2 - nx = 2n^2$ for x; for n.
3. Solve $x^2 - ax + b = 0$ for x.
4. Solve $v^2 + 2sv + t = 0$ for v.
5. Solve $m^2 - am - bm + ab = 0$ for m.
6. Solve $t^2 - (2m - n)t = 2mn$ for t.
7. Solve $y^2 + 2b + b^2 = 2by + 2y$ for y.
8. Solve $P^2 + 2P + 1 = nP^2 + nP$ for P.
9. Solve $\frac{a}{a-n} + \frac{n-a}{a} = 2$ for n.
10. Solve $\frac{R^2 + 2mR}{m-n} = \frac{m}{n} + \frac{R}{2n}$ for R.

11. Solve $\frac{at+b}{bt+a} - \frac{ct-d}{dt-c} = 0$ for t.

12. Solve $\frac{m}{r+m} + \frac{n}{r+n} + \frac{p}{r+p} = 0$ for r.

13. Solve $\left(\frac{V}{n+1} + n\right)^2 - \left(\frac{V}{n-1} - n\right)^2 = 0$ for V.

14. Solve $\frac{k^2}{d+N} - \frac{k^2}{d-N} = f$ for N.

15. The formula $E^2 = \frac{JWhR}{t}$ is used in electrical work. Solve for E.

16. $H = 0.24\,C^2Rt$ expresses a law of heat. Solve for C.

17. $E = \frac{4\,n^2L^2w}{g}$ expresses a law of sound. Solve for L.

18. $\frac{F}{f} = \frac{d^2}{D^2}$ expresses a law of attraction between two bodies. Solve for D.

19. The horse power of automobile engines is computed by the formula $H = KND(D-1)(R+2)$. Solve for D.

20. The elevation of a point above sea level is obtained by use of the formula $H = 513\,t + t^2$. Solve for t.

21. The area of a ring is computed by the formula

$$A = \pi(r^2 - r'^2).$$

Solve for r; for r'.

22. The volume of a frustum of a cone is computed by the formula $V = \frac{1}{3}H(R^2 + r^2 + Rr)$. Solve for R; for r.

92. Problems solved by Quadratics. — In solving problems by means of quadratic equations, since the roots of such equations may be fractions, negative numbers, surds, or even imaginary numbers, the nature of the roots obtained must be observed, and roots that do not conform to the requirements of the problems must be discarded.

EXERCISES

1. Separate 30 into two parts so that the square of one part exceeds the square of the other part by 160.

2. The hypotenuse and longer leg of a right triangle are together 18 inches long. The shorter leg is 6 inches long. Find the hypotenuse and longer leg.

3. The area of a rectangle is 192 square inches. The length is 4 inches more than the width. Find the length and width.

4. The area of a triangle is 240 square inches. The altitude is 4 inches less than the base. Find the altitude and base.

5. One side of a rectangle is 8 inches longer than the other. The diagonal is 40 inches. Find the area.

6. When the sides of a square are increased 4 inches, the area is increased 176 square inches. Find the side of the square before it is increased.

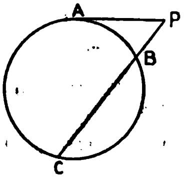
7. A rectangular piece of tin was twice as long as it was wide. A 4-inch square was cut from each corner and the sides and ends were turned up so as to make a box containing 1536 cubic inches. Find the dimensions of the piece of tin.

8. The length of a rectangular prism containing 240 cubic inches is 10 inches. Its entire surface is 248 square inches. Find the other two dimensions.

9. It is 180 rods around a rectangular field containing $11\frac{1}{4}$ acres. Find the dimensions.

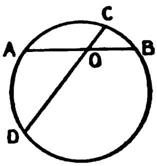
10. A room is 6 feet longer than it is wide. If the diagonal is 30 feet, find the length and width of the floor.

11. It is shown in geometry that if from a point P without a circle a tangent PA and a secant PC be drawn as in the margin, $\overline{PA}^2 = PB \times PC$. If $PA = 10$ and $BC = 4$, find PC.



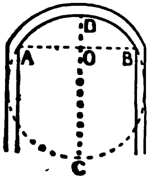
12. Using 4000 miles for the radius of the earth, and making use of the principle stated in Problem 11, find how far a man can

see over a smooth body of water when standing in a boat so that his eye is 10 feet above the level of the water.



13. It is shown in geometry that if two chords, AB and CD, intersect at O, $OA \times OB = OC \times OD$. If $OC = 2$ inches, $OD = 7$ inches, and $AB = 8$ inches, find the lengths of OA and OB.

14. A window 5 feet wide is to be surmounted by a circular stone arch, cut to an arc of a 12-foot circle. Find the rise by use of the principle stated in Problem 13.



SUGGESTION. — $OD =$ rise. $AO = OB$. $DC = 12$ feet.

15. A farmer has a field of wheat 40 rods wide and 80 rods long. How wide a strip must be cut around the field in order to have one half of it cut?

16. A tree 100 feet high is broken off by a storm so that the top reaches the ground 30 feet from the foot of the tree. Find the height of the part standing.

17. By increasing the edges of a cube each 2 inches, the volume was increased 1016 cubic inches. What was the length of the cube before it was increased?

18. A rectangular field is 3 times as long as it is wide. If its width is increased by 20 rods, its area is doubled. Find the area of the field.

19. A farmer sold his crop of oats for $600. If he had held them for 4 months, he could have gotten 10 cents per bushel more for them, in which case he could have received as much for them and have had 300 bushels left. How many bushels had he?

20. A man bought a farm for $12,600. He then sold all but 40 acres of it at a gain of $20 per acre over cost, receiving the entire cost of the farm. How many acres did he sell?

21. A party engaged a yacht for two weeks for $168. Had there been two less, the cost to each would have been $2 more. How many were there?

22. A tree was broken over by a storm so that the top touched the ground 50 feet from the foot of the stump. The stump was $\frac{3}{8}$ of the height of the tree. What was the height of the tree?

23. A tree which stood at the edge of the bank of a stream fell with its top in the water. The tree was 60 feet high, the bank on which it stood was 15 feet above the water level, and the body of the tree passed under the surface of the water at a point 20 feet from the bank. What portion of the tree was under water?

24. The hypotenuse of a right triangle is 4 inches longer than one leg and 2 inches longer than the other. Find the sides of the triangle.

25. One pipe can fill a cistern in 6 minutes less time than is required for another pipe to fill it. The two together can fill it in $10\frac{2}{7}$ minutes. Find the time required for each pipe alone to fill the cistern.

26. One of two pipes can fill a tank in 28 minutes, and the time required for the other pipe to fill it is $19\frac{4}{9}$ minutes longer than is required for the two pipes together to fill it. Find the time required for the two pipes together to fill the tank.

27. Two trains on the same road start at the same time from stations 225 miles apart. One takes $\frac{2}{3}$ minute longer than the other to run a mile, and they meet in 3 hours. Find the speed of each train.

28. A teamster having 12 miles to drive increased his speed one mile an hour after he had traveled 2 miles. He thus finished the distance in half an hour less time than it would have taken had he not increased his speed. How long did it take to drive the 12 miles?

SUGGESTION. — Let his first rate be x miles an hour.

29. A merchant bought a certain number of mirrors for $36, and, after breaking one, sold the rest for 50 cents apiece more than they cost him, thus making $2.50 by the transaction. How many did he buy?

30. The rate at which a man can row in still water is twice the rate of the current of a river. He rows 6 miles down the river and back in 4 times as many hours as he could row miles per hour in still water. Find the rate of the current and the rate of his rowing in still water.

31. The number of square inches in the area of a square exceeds the number of inches in its perimeter by 32. What is its area?

32. The side of a square is the same as the diameter of a circle. The area of the square exceeds that of the circle by approximately 3.4336 square inches. Find the diameter of the circle.

33. The base of a triangle exceeds its altitude by 2 inches, and its area is 112 square inches. Find its altitude.

34. What is the area of a square field whose diagonal is 10 rods longer than one of its sides?

35. By adding a border to a rug 6 feet by 9 feet its area is increased 34 square feet. Find the width of the border.

93. Character of the Roots of a Quadratic. — In the use of the formula

$$x = \frac{-B \pm \sqrt{B^2 - 4AC}}{2A},$$

in § 90, it was observed that the character of the roots depended upon the expression under the radical sign. This expression, $B^2 - 4AC$, for that reason, is called the **discriminant** of the equation $Ax^2 + Bx + C = 0$.

A study of the formula shows that:

(1) *When the discriminant is positive and not a perfect square, the roots are real, unequal, and involve surds.*

(2) *When the discriminant is positive and a perfect square, the roots are real, unequal, and free of surds.*

(3) *When the discriminant is zero, the roots are real and equal.*

(4) *When the discriminant is negative, the roots are imaginary.*

From these observations we can determine the character of the roots of any given quadratic equation, when desired, without solving the equation.

Example. — Determine the nature of the roots of $P^2 + 2P + 5 = 0$.

Here $A = 1$, $B = 2$, $C = 5$. Hence, $B^2 - 4AC = -16$. Hence, the roots are imaginary.

EXERCISES

By use of the discriminant, determine the character of the roots of the following:

1. $n^2 - 5n + 4 = 0$.
2. $a^2 + 6a - 1 = 0$.
3. $t^2 + t + 3 = 0$.
4. $3x^2 + 2x - 1 = 0$.
5. $2A^2 - 7A + 8 = 0$.
6. $12p^2 + 5p + 2 = 0$.
7. $3V^2 = 4V - 2$.
8. $9r^2 = 4r + 5$.
9. $4x^2 = 12x - 9$.
10. $7x - 11 = 4x^2$.
11. $2K^2 + 9K = 4$.
12. $16b^2 + 25 = 40b$.
13. $R^2 - 6R + 6 = 0$.
14. $n(n - 4) = n - 2$.

For what values of c will the following have equal roots?

15. $cx^2 + 3x - 5 = 0$.

Solution. — The discriminant is $9 + 20c$. For equal roots, $9 + 20c = 0$. Solving, $c = -\frac{9}{20}$.

16. $cx^2 - 6x = 4$.
17. $4x^2 + cx + 25 = 0$.
18. $2t^2 + ct + 32 = 0$.
19. $3A^2 - 2A + c = 0$.
20. $2n^2 + 8n + c = 0$.
21. $3V^2 + 10V + c + 3 = 0$.

Determine the relation between m and n for which the following will have equal roots:

22. $mx^2 + 5x + n = 0$.
23. $my^2 - ny + 4 = 0$.
24. $t^2 + mt + n = 0$.
25. $nP^2 - 2mP + 8 = 0$.

94. Relation between the Roots and Coefficients. — By dividing both members of the general quadratic $Ax^2 + Bx + C = 0$ by A, it may be written in the form $x^2 + px + q = 0$.

Solving this equation by the formula, and calling the roots r_1 and r_2, we get

$$r_1 = \frac{-p + \sqrt{p^2 - 4q}}{2} \text{ and } r_2 = \frac{-p - \sqrt{p^2 - 4q}}{2}.$$

Adding these roots, $\quad r_1 + r_2 = -p.$

Multiplying them, $\quad r_1 r_2 = q.$

Hence, if an equation is written in the form $x^2 + px + q = 0$,

(1) *The sum of the roots is the coefficient of x with its sign changed;*

(2) *The product of the roots is the term free of x.*

The principles (1) and (2) enable one to write down the quadratic equation whose roots are given.

EXAMPLE. — Form the equation whose roots are 7 and -3.

The sum of the roots is $7 - 3$, or 4, and their product is -21.

Hence, the equation is $x^2 - 4x - 21 = 0$.

In testing the roots of an equation to see if the solution is correct, especially in case the roots obtained are imaginary or involve surds, so that substitution of the roots in the original equation is not convenient, it is advantageous to use principles (1) and (2).

EXAMPLE. — Solve and check $7y^2 - 2y + 3 = 0$.

By the formula, $\quad y = \frac{1}{7} \pm \frac{1}{7}\sqrt{-20}.$

The sum of these roots is $\frac{2}{7}$. Their product is $(\frac{1}{7})^2 - (\frac{1}{7}\sqrt{-20})^2$, or $\frac{3}{7}$. Hence, since the equation may be written in the form $y^2 - \frac{2}{7}y + \frac{3}{7} = 0$, the roots check.

EXERCISES

Form the equations whose roots are the following:

1. 6, 2.
2. 9, 1.
3. -5, -3.
4. 8, -2.
5. -7, 6.
6. -12, -2.

7. $10, --5$.
8. $-3, -20$.
9. $9, -11$.
10. $\frac{1}{2}, \frac{1}{3}$.
11. $\frac{4}{5}, -\frac{1}{5}$.
12. $\frac{3}{8}, -\frac{1}{3}$.
13. $-\frac{1}{4}, -\frac{5}{6}$.
14. $a, 3a$.
15. $-2n, 5m$.
16. $1+\sqrt{5}, 1-\sqrt{5}$.
17. $-4+\sqrt{3}, -4-\sqrt{3}$.
18. $1+\sqrt{-6}, 1-\sqrt{-6}$.
19. $-3+\sqrt{-4}, -3-\sqrt{-4}$.

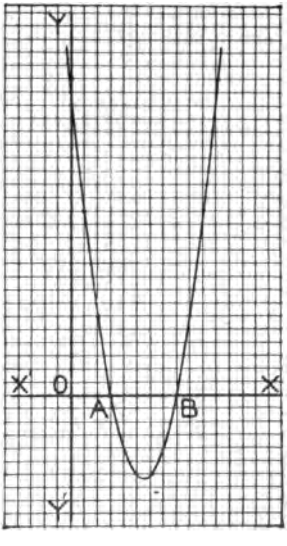
Solve and check by use of principles (1) and (2):

20. $n^2+n-6=0$.
21. $t^2-5t+7=0$.
22. $3a^2-2a-5=0$.
23. $5V^2=V-1$.
24. $N(N-3)=4$.
25. $2r^2-5(r-5)=0$.
26. $(A+3)(A-2)=3$.
27. $x^2-ax+3b=0$.

95. Graphical Interpretation of Roots. — Any value of x which satisfies both equations of the system

$$\begin{cases} y = Ax^2 + Bx + C, & (1) \\ y = 0 & (2) \end{cases}$$

satisfies the equation $Ax^2+Bx+C=0$.



The graph of (1) is some kind of curve, and the graph of (2) is the x-axis (see § 51). Hence, since the coördinates of the points where the graphs of (1) and (2) cross must satisfy both (1) and (2) (see § 53), the roots of $Ax^2+Bx+C=0$ may be obtained by observing the x-distances of the points where the graph of (1) crosses the x-axis.*

---

* For another method of solving a quadratic graphically, see FIRST COURSE, pages 292 and 293.

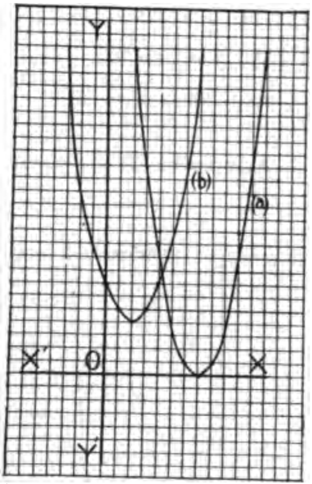
Example. — Solve graphically $x^2 - 11x + 24 = 0$.

Let
$$y = x^2 - 11x + 24.$$

By assigning values to x, and computing the corresponding values of y, the following sets of values are obtained:

x	0	1	2	3	5	7	10	11
y	24	14	6	0	-6	-4	14	24

Locating the points whose coördinates are these sets of values of x and y, and connecting them by a smooth line, gives the curve on the preceding page.



It is seen that this curve crosses the x-axis at two points, A and B, whose x-distances are 3 and 8, respectively. Hence, the roots of $x^2 - 11x + 24 = 0$ are 3 and 8.

If the roots of a quadratic are *equal*, the graph of (1) will just touch the x-axis at one point.

Example. — The roots of $x^2 - 14x + 49 = 0$ are each $+7$. The graph of $y = x^2 - 14x + 49$ is the curve (a) in the figure, which just touches the x-axis at a point whose x-distance is the root $+7$.

If the roots of a quadratic are *imaginary*, the graph of (1) will not cross or touch the x-axis at all.

Example. — The roots of $x^2 - 4x + 8 = 0$ are $2 \pm 2i$. The graph of $y = x^2 - 4x + 8$ is the curve (b) in the figure, which lies entirely above the x-axis.

EXERCISES

Solve graphically:

1. $x^2 - 12x + 35 = 0$.
2. $x^2 + x - 72 = 0$.
3. $x^2 + 14x + 48 = 0$.
4. $2x^2 - 23x + 56 = 0$.
5. $3x^2 + 14x = 40$.
6. $5x^2 + 3x - 110 = 0$.

Find, by the discriminant, the nature of the roots of the following, and give their graphical interpretation:

7. $4x^2 - 33x + 8 = 0.$
8. $2x^2 + 3x = 54.$
9. $x^2 + 16x + 64 = 0.$
10. $x^2 + x + 5 = 0.$
11. $9x^2 - x + 2 = 0.$
12. $4x^2 + 25 = 20x.$

SUPPLEMENTARY EXERCISES

If r_1 and r_2 are the roots of $Ax^2 + Bx + C = 0$, then

$$x^2 + \frac{B}{A}x + \frac{C}{A} = x^2 - (r_1 + r_2)x + r_1r_2 = (x - r_1)(x - r_2),$$

by § 94. Hence,

$$Ax^2 + Bx + C = A(x - r_1)(x - r_2).$$

Therefore, the three factors of an expression of the form $Ax^2 + Bx + C$ are A, $x - r_1$, and $x - r_2$, where r_1 and r_2 are the roots of the equation formed by putting $Ax^2 + Bx + C$ equal to zero.

Hence any trinomial of the form $Ax^2 + Bx + C$ may be factored, if factors containing surds or imaginary numbers are accepted. It sometimes is desirable to find such factors of a trinomial.

Evidently, by use of the discriminant $B^2 - 4AC$ we can tell whether the factors of such a trinomial are *real* or *imaginary*, or *free of surds*.

Find the nature of the factors of:

1. $x^2 - 15x + 54.$
2. $24t^2 - t - 10.$
3. $3n^2 - 9n + 28.$
4. $2y^2 + y + 3.$
5. $2V^2 + 5V - 7.$
6. $x^2 - 9x + 3.$
7. $4x^2 + 2x + 1.$
8. $6a^2 + 11a - 40.$
9. $10R^2 - R + 2.$
10. $6E^2 - E - 35.$

Factor:

11. $2x^2 + 2x - 5.$

SOLUTION. — Let $2x^2 + 2x - 5 = 0$.

Solving, $$x = -\tfrac{1}{2} \pm \tfrac{1}{2}\sqrt{11}.$$

Hence, $$2x^2 + 2x - 5 = 2(x + \tfrac{1}{2} + \tfrac{1}{2}\sqrt{11})(x + \tfrac{1}{2} - \tfrac{1}{2}\sqrt{11})$$

12. $2s^2 - s + 2$.
13. $5P^2 + 3P + 4$
14. $3b^2 - b - 1$.
15. $9d^2 + 4d + 1$.
16. $n^2 + 2n - 143$.
17. $10y^2 + 11y - 8$.
18. $7A^2 - A + 16$
19. $2T + 15 + T^2$

CHAPTER VIII

HIGHER EQUATIONS

96. Higher Equations. — In preceding chapters we have worked out general methods of solving any linear or any quadratic equation in one unknown number. General algebraic methods of solving any equation of the third or of the fourth degree also have been worked out by mathematicians. But in *elementary algebra* we shall undertake to solve only two special kinds of equations of degree higher than the second.

97. Higher Equations solved by Factoring. — Some equations of higher degree than the second may be solved by factoring. The method is the same as that of solving quadratics by factoring.

EXAMPLE 1. — Solve $n^3 + 1 = 0$.

Factoring, $(n + 1)(n^2 - n + 1) = 0.$

This equation is satisfied by any values of n that will make either of the factors $n + 1$ or $n^2 - n + 1$ equal to zero, for then their product would be zero. Hence, equating each factor to zero, we get

$$n + 1 = 0 \text{ and } n^2 - n + 1 = 0.$$

Solving, $n + 1 = 0,\ n = -1.$

Solving, $n^2 - n + 1 = 0,\ n = \frac{1}{2} \pm \frac{1}{2}\sqrt{-3}.$

EXAMPLE 2. — Solve $t^3 = 64$.

Transposing, $t^6 - 64 = 0.$

Factoring, $(t + 2)(t - 2)(t^2 + 2t + 4)(t^2 - 2t + 4) = 0.$

Putting each factor equal to zero,

$$t+2=0,\ t-2=0,\ t^2+2t+4=0,\ t^2-2t+4=0.$$

Solving each of these,

$$t=-2,\ 2,\ -1\pm\sqrt{-3},\ \text{or}\ 1\pm\sqrt{-3}.$$

Example 3. — Solve $A^3+5A=6A^2-12$.

Transposing, $A^3-6A^2+5A+12=0.$

Factoring by remainder theorem,

$$(A+1)(A-3)(A-4)=0.$$

Putting each factor equal to zero, and solving,

$$A=-1,\ 3,\ \text{or}\ 4.$$

As appears in each of the above examples, so it may be shown in general that

The number of roots of an equation is equal to the degree of the equation.

To solve any equation in one unknown number by factoring,

(1) *Transpose all terms to the left member, making the right member zero.*

(2) *Factor the resulting left member, and equate each factor to zero.*

(3) *Solve the resulting equations.*

EXERCISES

Solve:

1. $n^3-1=0$.
2. $a^6-1=1$.
3. $V^3=27$.
4. $x^4-1=0$.
5. $y^4-81=0$.
6. $t^4-5t^2+4=0$.
7. $k^4-16=0$.
8. $N^4-2N^2=15$.
9. $8s^3-27=0$.
10. $64A^3+125=0$.

11. $2g^4 = 5g^2 + 12.$

12. $m^6 + 7m^3 = 8.$

13. $4E^4 = 20E^2 - 21.$

14. $9c^4 + 18c^2 + 8 = 0.$

15. $x^3 + 2x^2 = x + 2.$

16. $z^3 - z^2 - 4z + 4 = 0.$

17. $T^3 + T = 3T^2 + 3.$

18. $a^3 + 5a^2 = 4a + 20.$

19. $R^4 + 4R^3 + R + 4 = 0.$

20. $h^5 + h^3 = h^2 + 1.$

21. $3P^4 + P^3 + 2P^2 + 24P = 0.$

22. $n^4 - 3n^3 + 3n^2 - n = 0.$

23. Find the three cube roots of 64.

SUGGESTION. — Let n be the cube root of 64.

Then $n^3 = 64.$

24. Find the four fourth roots of 81.

25. Find the six sixth roots of 1.

26. Find the six sixth roots of 64.

98. Higher Equations written in the Form of Quadratics. — Some equations of higher degree than the second may be written in the form of quadratics, and solved by the methods of solving quadratics.

EXAMPLE 1. — Solve $r^4 - 13r^2 + 36 = 0.$

This has the form of a quadratic in r^2. Solving for r^2 by the formula,

$$r^2 = \frac{13 \pm \sqrt{169 - 144}}{2}$$

$$= 9 \text{ or } 4.$$

Now, solving, $r^2 = 9,\ r = \pm 3.$

Solving, $r^2 = 4,\ r = \pm 2.$

Hence, the four roots are 2, − 2, 3, − 3.

EXAMPLE 2. — Solve $\dfrac{k^2}{k+1} + \dfrac{2(k+1)}{k^2} = 3.$

Putting n for $\dfrac{k^2}{k+1}$, the equation may be written

$$n + \frac{2}{n} = 3.$$

Solving, $n = 1 \text{ or } 2.$

Hence, the given equation is equivalent to the equations

$$\frac{k^2}{k+1} = 1, \text{ and } \frac{k^2}{k+1} = 2.$$

Solving $\frac{k^2}{k+1} = 1,$ $k = \frac{1}{2} \pm \frac{1}{2}\sqrt{5}.$

Solving $\frac{k^2}{k+1} = 2,$ $k = 1 \pm \sqrt{3}.$

EXAMPLE 3. — Solve $2v^2 + 2v + 1 = \frac{10}{v^2 + v}.$

This may be written $2(v^2 + v) + 1 = \frac{10}{v^2 + v}.$

Putting n for $v^2 + v$,

$$2n + 1 = \frac{10}{n}.$$

Solving, $n = 2 \text{ or } -\frac{5}{2}.$

Hence, the given equation is equivalent to the equations

$$v^2 + v = 2 \text{ and } v^2 + v = -\frac{5}{2}.$$

Solving these, $v = 1,\ -2,\ -\frac{1}{2} + \frac{3}{2}\sqrt{-1},\ -\frac{1}{2} - \frac{3}{2}\sqrt{-1}.$

EXERCISES

Solve:

1. $x^4 + 2x^2 = 48.$
2. $6a^4 + a^2 = 15.$
3. $21V^4 = V^2 + 10.$
4. $n^6 + 9n^3 + 8 = 0.$
5. $8p^6 = 37p^3 + 216.$
6. $y^8 - 2y^4 + 1 = 0.$
7. $(x^2 - 1)^2 + 24 = 11(x^2 - 1).$
8. $(t^2 + 2t)^2 - 2(t^2 + 2t) = 3.$
9. $(n^2 - n)^2 - 8(n^2 - n) + 12 = 0.$
10. $(2P^2 + 7P)^2 - 3(2P^2 + 7P) = 10.$

11. $\dfrac{x}{1-x^2}+\dfrac{2-2x^2}{x}=3.$

12. $\dfrac{m}{2m^2+1}+\dfrac{2m^2+1}{m}=2.$

13. $\dfrac{2d+1}{d^2}+\dfrac{d^2}{2d+1}=2.$

14. $A^2+1=\dfrac{4}{A^2+1}.$

15. $c^2+c-1=\dfrac{2}{c^2+c}.$

16. $H^2+3H=1-\dfrac{1}{H^2+3H+1}.$

17. $4n^2-10n+3=\dfrac{1}{2n^2-5n+1}.$

18. $(2v^2+6v)^2=1-(v^2+3v-2).$

19. $(R^2-R+4)^2+(R^2-R)=2.$

20. $(b^2-3b+1)(b^2-3b+2)=12.$

21. $(W^2+5W-1)(W^2+5W+1)+1=0.$

22. $\left(x+\dfrac{8}{x}\right)^2+\left(x+\dfrac{8}{x}\right)=42.$

23. $\left(n-\dfrac{5}{n}\right)^2-\left(n-\dfrac{5}{n}\right)=30.$

24. $a^2+a+\dfrac{1}{a}+\dfrac{1}{a^2}=4.$

Suggestion. — $a^2+2+\dfrac{1}{a^2}=\left(a+\dfrac{1}{a}\right)^2.$

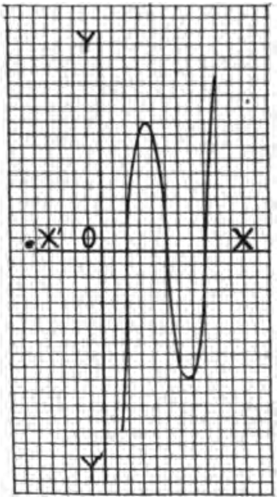
25. $y^4+3y^3-2y^2-3y+1=0.$

Suggestion. — **Divide by y^2, and arrange as in Ex. 24.**

26. $D^4-3D^3+4D^2-3D+1=0.$

SUPPLEMENTARY EXERCISES

The method of § 95 may be used to represent graphically the roots of an equation of any degree in one unknown number.



Represent graphically the roots of:

1. $x^3 - 15x^2 + 66x - 80 = 0.$

Solution. — Let $y = x^3 - 15x^2 + 66x - 80.$ By assigning values to x, computing the corresponding values of y, locating the points whose coördinates are the sets of values thus found, and joining the consecutive points by a smooth line, the curve shown in the figure is obtained.

This curve cuts the x-axis at three points, whose x-distances are the roots 2, 5, and 8, respectively.

2. $x^3 - 15x^2 + 71x - 105 = 0.$

3. $x^3 - 31x - 30 = 0.$

4. $x^4 - 29x^2 + 100 = 0.$

5. $x^4 - 2x^3 - 72x^2 + 128x + 512 = 0.$

Determine graphically the nature of the roots of:

6. $x^3 - 6x^2 + 16x - 96 = 0.$

7. $x^3 - 22x^2 + 145x - 300 = 0.$

CHAPTER IX

EQUATIONS INVOLVING SURDS

99. Equations involving Surds. — To solve an equation in which the unknown number is involved in one or more surds, it is necessary first to free the unknown number of radical signs. To do this, both members of the equation must be raised to the same power.

EXAMPLE. — Solve $\sqrt{n+2}+\sqrt{3-n}=3.$

Transposing, $\sqrt{n+2}=3-\sqrt{3-n}.$

Squaring both members, $n+2=9-6\sqrt{3-n}+3-n.$

Transposing and combining, $2n-10=-6\sqrt{3-n}.$

Dividing by -2, $5-n=3\sqrt{3-n}.$

Squaring both members, $25-10n+n^2=27-9n.$

Transposing and combining, $n^2-n-2=0.$

Solving, $n=2 \text{ or } -1.$

The above example illustrates the following general rule for freeing an equation of radical signs:

(1) *By transposing, if necessary, change the equation into an equivalent one of which one member consists of a single surd.*

This is called **isolating the surd.**

(2) *Raise both members to such a power as will free this isolated surd of the radical sign.*

(3) *If surds still remain, repeat the operation.*

In checking the solution of an equation involving surds by substituting the values of the roots found, it must be remembered that *only the principal roots of the radicals involved may be used.* (See § 64.)

Thus, 5 is not a root of $\sqrt{3t+1}+\sqrt{t-1}=2$, although when $t=5$, it would become $4-2=2$ if the negative square root of $t-1$ were taken for the second term. The equation of which 5 is a true root is $\sqrt{3t+1}-\sqrt{t-1}=2$.

100. Extraneous Roots. — When both members of an equation are raised to the same power, roots may be introduced (extraneous roots) that will not satisfy the given equation.

For, let any equation be represented by $A=B$.

Squaring both members, $A^2=B^2$.

Transposing and factoring, $(A-B)(A+B)=0$.

This equation is equivalent to $A-B=0$ and $A+B=0$.

Of these two equations, $A-B=0$ alone is equivalent (see § 44) to the given equation. Hence, all of the roots of $A+B=0$ are introduced by squaring.

Hence, in solving an equation which involves surds, it is necessary to test all roots obtained and to discard those which do not satisfy the equation in its original form.

Example. — Solve $\sqrt{x+1}-\sqrt{8x}+5=0$. (1)

Isolating a surd, $\sqrt{x+1}=\sqrt{8x}-5$.

Squaring, $x+1=8x-10\sqrt{8x}+25$. (2)

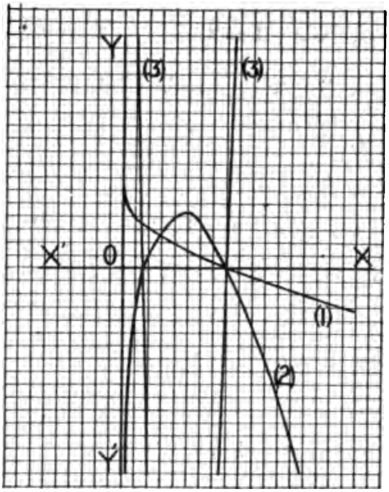
Isolating the surd, $10\sqrt{8x}=7x+24$.

Squaring, $800x=49x^2+336x+576$. (3)

Solving, $x=1\frac{23}{49}$ or 8.

By substitution it is seen that 8 satisfies (1), but $1\frac{23}{49}$ does not. The extraneous root $1\frac{23}{49}$ was introduced by squaring.

What took place in the preceding solution is shown graphically in the accompanying figure. The graphs of equations (1), (2), and (3) are the curves (1), (2), and (3), respectively (the two lines marked (3) are parts of one continuous curve that turns upward at a point 522+ spaces below the x-axis). Squaring equation (1) changed the curve (1) into curve (2), and squaring equation (2) changed the curve (2) into curve (3). Curve (1) shows that equation (1) has only one root, 8. Curve (2) shows that equation (2) has, in addition to the root 8, another root between 1 and 2. And curve (3) shows that equation (3) has the same roots as equation (2).



An equation involving surds may not be satisfied by any value of the unknown number. Such an equation is called an **impossible equation.**

Example. — Solve $\sqrt{2s+8}+2+2\sqrt{s+5}=0$.

Isolating a surd, $\sqrt{2s+8}+2=-2\sqrt{s+5}$.

Squaring, $2s+8+4\sqrt{2s+8}+4=4s+20$.

Hence, $2\sqrt{2s+8}=s+4$.

Squaring again, $8s+32=s^2+8s+16$.

Solving, $s=4$ or -4.

Neither of these roots satisfies the given equation.

EXERCISES

Solve:

1. $\sqrt{x+1}=2.$

2. $8-\sqrt{y-1}=6.$

3. $\sqrt{t+3}=8.$

4. $\sqrt{2x-3}=5.$

5. $5+\sqrt{n+7}=n.$

6. $\sqrt[3]{13+a}=2.$

7. $\sqrt[3]{2v-1}=3.$

8. $4\sqrt{m+5}=3\sqrt{3m+4}.$

9. $5\sqrt{R+2}-3\sqrt{4R-3}=0.$

10. $\sqrt{k+2}-1=\sqrt{k-3}.$

11. $\sqrt{15+3D}+2=\sqrt{23-D}.$

12. $2\sqrt{9+x}=3\sqrt{x+24}-10.$

13. $\sqrt{M^2+4M+9}=M+7.$

14. $\sqrt{5t}+4=\sqrt{12t+21}.$

15. $\sqrt{A}=\sqrt{A+25}-1.$

16. $\sqrt{d-12}=\sqrt{d}-2.$

17. $\sqrt{s}-7+\sqrt{s+7}=0.$

18. $2\sqrt{n+2}=\sqrt{n}.$

19. $w-3=\sqrt{2w^2-w+10}-5.$

20. $\sqrt{n+1}+\sqrt{n-2}=0.$

21. $\sqrt{h^2-7}=7-h.$

22. $\sqrt{r}+\sqrt{32+r}+16=0.$

23. $\sqrt{y+4}+\sqrt{y}+2=0.$

24. $\sqrt{n+2}-\sqrt{3-n}=3.$

25. $\sqrt{a-1}+\sqrt{3a+1}=2.$

26. $1+\sqrt{z+2}=\sqrt{z}.$

27. $\sqrt{2p-7}+\sqrt{2p+9}=8.$

28. $\sqrt[4]{a^2-5a+6}=\sqrt{a+4}.$

29. $\sqrt{2x-5}=\sqrt[4]{x^2-7x+12}.$

30. $\sqrt[3]{t+1}=\sqrt{t+1}.$

31. $\sqrt{y^2+1}=\sqrt[3]{y^2+1}.$

32. $\sqrt{x}-\sqrt{x-3}=\dfrac{2}{\sqrt{x}}.$

33. $\dfrac{2}{\sqrt{x}}=\sqrt{x}+\sqrt{x-1}.$

34. $\sqrt{n}+\sqrt{n-6}=\dfrac{3}{\sqrt{n-6}}.$

35. $\sqrt{a}+\sqrt{1+a}=\dfrac{2}{\sqrt{1+a}}.$

36. $\sqrt{s}-\sqrt{s-8}=\dfrac{2}{\sqrt{s-8}}.$

37. $\dfrac{20}{\sqrt{15+k}}-\sqrt{k}=\sqrt{15+k}.$

38. $\dfrac{12}{\sqrt{R^2+8}}=\sqrt{R^2+8}+R.$

39. $\dfrac{\sqrt{1+n}+\sqrt{1-n}}{\sqrt{1+n}-\sqrt{1-n}}=2.$

40. $\dfrac{1}{1-\sqrt{a}}-\dfrac{1}{1+\sqrt{a}}-\dfrac{8}{1-a}=0.$

41. $\frac{\sqrt{A+16}}{\sqrt{4-A}}+\frac{\sqrt{4-A}}{\sqrt{A+16}}-2\frac{1}{2}=0.$

42. $\sqrt{5N+2\sqrt{N^2-N+7}}-4=0.$

43. $\sqrt{13+3\sqrt{r+5+4\sqrt{r+1}}}-5=0.$

44. $3x^2-4x+\sqrt{3x^2-4x-6}=18.$

SUGGESTION. — Let $\sqrt{3x^2-4x-6}=y.$

Then $y^2+6+y=18.$

45. $\sqrt{n^2+3n+1}=1-2n^2-6n.$

46. $2\sqrt{4t^2-10t-2}=10t+10-4t^2.$

47. $r+22=3r^2-6\sqrt{3r^2-r-6}.$

48. $\sqrt{2a-4}+\sqrt{3a+4}-\sqrt{10a-4}=0.$

49. $\sqrt{2a-4}+\sqrt{3a+4}+\sqrt{10a-4}=0.$

50. $t=\pi\sqrt{\frac{l}{g}}$ expresses a law of the vibration of pendulums. Solve for l.

51. $v=\sqrt{2gs}$ gives the velocity acquired by a falling body. Solve for s.

52. $\frac{N}{n}=\frac{\sqrt{T}}{\sqrt{t}}$ expresses a law of vibration of strings, in musical instruments. Solve for t; for T.

53. $t=\frac{u-\sqrt{u^2-2gs}}{g}$ expresses a law of motion of projectiles fired vertically. Solve for s.

SUPPLEMENTARY EXERCISES

Solve the following, and show graphically the introduction of extraneous roots, as in the first example of § 100:

1. $\sqrt{x-1}+\sqrt{3x+1}=2.$
2. $\sqrt{3x+1}+\sqrt{x-1}=6.$
3. $\sqrt{2x+4}-\sqrt{x+3}=1.$
4. $\sqrt{4x}-\sqrt{2x+1}=1.$

CHAPTER X

SYSTEMS INVOLVING QUADRATIC OR HIGHER EQUATIONS

101. Systems involving Quadratic or Higher Equations. — The solution of a system of two equations in two unknown numbers in which one or both of the equations is of the second or higher degree cannot always be obtained by the methods of elementary algebra. Consequently in this chapter we shall only undertake to solve certain special types of systems.

As in the case of a system of two linear equations, to solve a system involving higher equations requires first the *elimination* of one of the unknown numbers. The method of elimination adapted to any given system depends upon the character of the equations in the system.

102. One Equation Linear and One Quadratic. — If one equation of a system is *linear* and the other *quadratic*, either unknown number may be eliminated by **substitution**.

EXAMPLE. — Solve $\begin{cases} m - 2n = 6, & (1) \\ m^2 + n^2 = 17. & (2) \end{cases}$

Solving (1) for m, $\qquad m = 6 + 2n.$

Substituting $6 + 2n$ for m in (2), $36 + 24n + 5n^2 = 17.$

Solving this quadratic, $\qquad n = -1$ or $-\frac{19}{5}$

Substituting -1 for n in (1), $\qquad m + 2 = 6.$

Hence, $\qquad m = 4.$

Substituting $-\frac{19}{5}$ for n in (1), $\qquad m + \frac{38}{5} = 6.$

Hence, $\qquad m = -\frac{8}{5}.$

Therefore there are two solutions:

$$m = 4,\ n = -1;\ \text{and}\ m = -\tfrac{8}{5},\ n = -\tfrac{12}{5}.$$

To solve a system containing one linear and one quadratic equation:

(1) *Solve the linear equation for one unknown number, expressed in terms of the other.*

(2) *Substitute the value found in the quadratic equation, and solve the resulting quadratic in one unknown number.*

(3) *Substitute each value found for the one unknown number in the linear equation, and solve for the corresponding value of the other.*

It is evident that:

Any system of one linear and one quadratic equation has two solutions.

EXERCISES

Solve:

1. $\begin{cases} r - s = 1, \\ r^2 + s^2 = 5. \end{cases}$

2. $\begin{cases} h + k = 6, \\ h^2 + k^2 = 20. \end{cases}$

3. $\begin{cases} a - 2b = 3, \\ a^2 - b^2 = 24. \end{cases}$

4. $\begin{cases} 4x + y = 7, \\ 2x^2 + xy = 5. \end{cases}$

5. $\begin{cases} m + n = 9, \\ m^2 + mn + n^2 = 67. \end{cases}$

6. $\begin{cases} x^2 - xy + y^2 = 12, \\ 3x = y + 10. \end{cases}$

7. $\begin{cases} Rr + 10 = 0, \\ r + 4R = 6. \end{cases}$

8. $\begin{cases} vt + 40 = 0, \\ 2v = 3t + 3. \end{cases}$

9. $\begin{cases} 2P + 2N = 5PN, \\ 2N + 2P = 5. \end{cases}$

10. $\begin{cases} a^2 + b^2 + 4a + 6b = 40, \\ a = 10 + b. \end{cases}$

11. $\begin{cases} 3V = 2T, \\ 3VT + 2V + T = 485. \end{cases}$

12. $\begin{cases} 2x^2 + 3y^2 = 11 + 4xy, \\ x - 5 = 3y. \end{cases}$

13. $\begin{cases} (6 - m)(7 - n) = 80, \\ m + n = 5. \end{cases}$

14. $\begin{cases} k-3=n, \\ k^2+19+n^2=3\,kn. \end{cases}$

15. $\begin{cases} x+y=4, \\ \dfrac{1}{x}+\dfrac{1}{y}=1. \end{cases}$

16. $\begin{cases} A+B=2, \\ \dfrac{3}{B}+\dfrac{2}{A}=6. \end{cases}$

17. $\begin{cases} p^2+pq+q^2=97, \\ 1-\dfrac{q}{p}=\dfrac{19}{p}. \end{cases}$

18. $\begin{cases} \dfrac{t}{u}+\dfrac{u}{t}=3\tfrac{1}{3}, \\ 3\,t-2\,u+12=0. \end{cases}$

19. $\begin{cases} 1-\dfrac{P}{Q}+\dfrac{P^2}{Q^2}=\dfrac{21}{Q^2}, \\ 1+\dfrac{P}{Q}=\dfrac{9}{Q}. \end{cases}$

Solve for x and y:

20. $\begin{cases} x-y=a, \\ xy=a^2+b^2. \end{cases}$

21. $\begin{cases} x+y=m, \\ 4\,xy=m^2-n^2. \end{cases}$

22. $\begin{cases} x-y=h, \\ xy=t^2. \end{cases}$

23. $\begin{cases} x+y=a, \\ x^2+y^2=b. \end{cases}$

24. In any circle whose radius is r, circumference c, and area S, $c=2\,\pi r$ and $S=\pi r^2$. By eliminating r between these two equations, find the equation between S and c.

103. Two Quadratic Equations, each of the Form $ax^2+by^2=c$. — If each equation of a system is a quadratic of the form $ax^2+by^2=c$, in which x and y are the unknown numbers, either unknown number may be eliminated by **addition or subtraction**.

EXAMPLE. — Solve
$$\begin{cases} x^2+2\,y^2=22, & (1) \\ 2\,x^2+y^2=17. & (2) \end{cases}$$

Multiplying (1) by 2, $2\,x^2+4\,y^2=44.$

Subtracting (2) from this, $3\,y^2=27.$

Solving, $y=\pm 3.$

Substituting 3 for y in (1), $x^2+18=22.$

Solving, $x=\pm 2.$

Substituting -3 for y in (1), $x^2+18=22.$

Hence, $x=\pm 2.$

Therefore there are four solutions:

$x=2,\ y=3$; $x=-2,\ y=3$; $x=2,\ y=-3$; $x=-2,\ y=-3.$

To solve a system of two quadratic equations, each of the form $ax^2 + by^2 = c$:

(1) *Eliminate one unknown number by addition or subtraction, as in a system of linear equations.*

(2) *Solve the resulting quadratic in one unknown number.*

(3) *Substitute each value of the unknown number thus found in either of the given equations, and find the corresponding values of the other unknown number.*

It is evident that for each of the two values found for the first unknown number there are two corresponding values of the other. Hence:

Any system of two quadratic equations, each of the form $ax^2 + by^2 = c$, *has four solutions.*

It has been observed that in the case of every system solved in this and the preceding sections,

The number of solutions is equal to the product of the degrees of the two equations.

It may be shown that this rule is true in general. If one equation is of the *m*th degree and the other of the *n*th degree, the number of solutions will be *mn*. Exceptions to this rule in certain peculiar systems will be noted later.

EXERCISES

Solve:

1. $\begin{cases} 3x^2 - 8k^2 = 40, \\ 5x^2 + k^2 = 81. \end{cases}$

2. $\begin{cases} 3A^2 - 2B^2 = 0, \\ 5A^2 - 3B^2 = 1. \end{cases}$

3. $\begin{cases} r^2 - s^2 = 7, \\ r^2 + s^2 = 25. \end{cases}$

4. $\begin{cases} 4m^2 + 9n^2 = 36, \\ m^2 + n^2 = 25. \end{cases}$

5. $\begin{cases} 4W^2 - 9V^2 = 36, \\ W^2 + V^2 = 16. \end{cases}$

6. $\begin{cases} 3x^2 - 5y^2 = 30, \\ x^2 + 5y^2 = 70. \end{cases}$

7. $\begin{cases} v^2 + 4t^2 = 40, \\ 2v^2 - 9t^2 = 63. \end{cases}$

8. $\begin{cases} R^2 - 2D^2 = 17, \\ 2R^2 + D^2 = 54. \end{cases}$

9. $\begin{cases} 4a^2 - 9b^2 = 0, \\ 6a^2 + b^2 = 3\frac{2}{9}. \end{cases}$

10. $\begin{cases} 4P^2 + 3Q^2 = 43, \\ 3P^2 - Q^2 = 3. \end{cases}$

11. $\begin{cases} 9x^2 + y^2 = 29, \\ 27x^2 - 2y^2 + 38 = 0. \end{cases}$

12. $\begin{cases} E^2 - 25R^2 = 20, \\ 25R^2 = 88 - 2E^2. \end{cases}$

13. $\begin{cases} 3C^2 = y^2 + 27, \\ C^2 - y^2 + 45 = 0. \end{cases}$

14. $\begin{cases} 2r^2 - 3r'^2 = 20, \\ r^2 + 5r'^2 = 36. \end{cases}$

15. $\begin{cases} \dfrac{x}{y} + \dfrac{y}{x} = \dfrac{5}{16xy}, \\ x - \dfrac{y^2}{x} = \dfrac{3}{16x}. \end{cases}$

104. Two Quadratic Equations, One Homogeneous. — A **homogeneous equation** is one all of whose terms are of the same degree in the unknown numbers.

Thus, $2v^2 - vt + t^2 = 0$ is homogeneous, every term being of the second degree.

The process of solving a system of two quadratic equations, one of which is homogeneous, may be reduced to the process of § 102.

Example 1. — Solve $\begin{cases} m^2 - 5mn + 6n^2 = 0, & (1) \\ m^2 - mn + n^2 = 12. & (2) \end{cases}$

Factoring (1), $(m - 2n)(m - 3n) = 0.$

Hence, (1) is equivalent to the two equations

$$m - 2n = 0, \tag{3}$$

and

$$m - 3n = 0. \tag{4}$$

Therefore the given system is equivalent to the two systems

$A \begin{cases} m - 2n = 0, \\ m^2 - mn + n^2 = 12; \end{cases}$ $B \begin{cases} m - 3n = 0, \\ m^2 - mn + n^2 = 12. \end{cases}$

The systems A and B are each solved by substitution, as in § 102.

The solutions of A are $m = 4$, $n = 2$ and $m = -4$, $n = -2$.

The solutions of B are $m = \frac{6}{7}\sqrt{21}$, $n = \frac{2}{7}\sqrt{21}$ and $m = -\frac{6}{7}\sqrt{21}$, $n = -\frac{2}{7}\sqrt{21}$.

Hence, these four solutions are the solutions of the given system.

Example 2. — Solve $\begin{cases} 5x^2 + 4xy - y^2 = 0, & (1) \\ x^2 + x + y = 5. & (2) \end{cases}$

Solving (1) for x by the quadratic formula,

$$x = \frac{-4y \pm \sqrt{16y^2 + 20y^2}}{10}$$
$$= -y \text{ or } \tfrac{1}{5}y.$$

Hence, the given system is equivalent to the two systems

$$A \begin{cases} x = -y, \\ x^2 + x + y = 5; \end{cases} \qquad B \begin{cases} x = \tfrac{1}{5}y, \\ x^2 + x + y = 5. \end{cases}$$

Solving system A by the method of § 102,

$$x = -3 + \sqrt{14},\ y = -15 + 5\sqrt{14};\ x = -3 - \sqrt{14},\ y = -15 - 5\sqrt{14}.$$

Similarly, the solutions of B are found to be

$$x = \sqrt{5},\ y = -\sqrt{5};\ x = -\sqrt{5},\ y = \sqrt{5}.$$

These four solutions are the solutions of the given system.

The method of Example 1 may be used only in case the first member of the homogeneous equation may be factored. The method of Example 2 may be used in case of any homogeneous equation, whether the first member can be factored or not, and hence is the more general method of the two.

To solve a system of two quadratic equations, one being homogeneous, find by factoring or by solving by the quadratic formula two linear equations to which the homogeneous equation is equivalent. Use each of these with the other given equation to form a new system. Solve these two systems by the method of § 102.

EXERCISES

Solve:

1. $\begin{cases} x^2 - 3xy + 2y^2 = 0, \\ 2x^2 + xy - y^2 = 9. \end{cases}$

2. $\begin{cases} a^2 - 5ab + 6b^2 = 0, \\ a^2 - b^2 = 27. \end{cases}$

3. $\begin{cases} 3m^2 - 2mn - n^2 = 0, \\ m + n + n^2 = 32. \end{cases}$

4. $\begin{cases} v^2 + 7vt + 10t^2 = 0, \\ v^2 + 3v + vt - 4t = 33. \end{cases}$

5. $\begin{cases} x^2 - y^2 = 0, \\ 2x^2 + xy = 48. \end{cases}$

6. $\begin{cases} 2V^2 + 3Vg = 0, \\ V^2 - Vg + g^2 = 19. \end{cases}$

7. $\begin{cases} 3PQ + 5Q^2 = 0, \\ 3P^2 + 4PQ = 15. \end{cases}$

8. $\begin{cases} a^2 + ab = 0, \\ a^2 - ab + b^2 = 27. \end{cases}$

105. Quadratics with All Unknown Terms of Second Degree. — A system of two quadratic equations of which neither is homogeneous, but in each of which all unknown terms are of the second degree, may be solved by first combining the given equations so as to obtain a homogeneous equation.

Example. — Solve $\begin{cases} p^2 - 7pq - 9q^2 = 9, & (1) \\ p^2 + 5pq + 11q^2 = 5. & (2) \end{cases}$

Evidently we may get a homogeneous equation by eliminating the terms free of unknown numbers.

Multiplying (1) by 5, $5p^2 - 35pq - 45q^2 = 45.$ (3)

Multiplying (2) by 9, $9p^2 + 45pq + 99q^2 = 45.$ (4)

Subtracting (3) from (4), $4p^2 + 80pq + 144q^2 = 0,$

or $p^2 + 20pq + 36q^2 = 0.$ (5)

Now the given system is equivalent to the new system formed of (5) and either (1) or (2). Let us form the system

$$\begin{cases} p^2 + 20pq + 36q^2 = 0, & (5) \\ p^2 + 5pq + 11q^2 = 5. & (2) \end{cases}$$

This system is solved by the method of § 104.
The solutions are:

$p = 2,\ q = -1;\quad p = -2,\ q = 1;\quad p = 2\frac{4}{7},\ q = -\frac{1}{7};\quad p = -2\frac{4}{7},\ q = \frac{1}{7}.$

By eliminating the known terms, derive from the given equations a homogeneous equation. Use this homogeneous equation with either of the given equations to form a new system. Solve this system by the method of § 104.

EXERCISES

Solve:

1. $\begin{cases} x^2 + xy = 21, \\ 2\,xy - y^2 = 8. \end{cases}$

2. $\begin{cases} 2\,m^2 - mn + n^2 = 16, \\ m^2 + mn + 2\,n^2 = 44. \end{cases}$

3. $\begin{cases} A^2 - 3\,AB + 2\,B^2 = 3, \\ 2\,A^2 + B^2 = 6. \end{cases}$

4. $\begin{cases} h^2 + k^2 = 10, \\ 3\,k^2 + hk = 6. \end{cases}$

5. $\begin{cases} w^2 + wv + v^2 = 4, \\ w^2 - 2\,wv = 12. \end{cases}$

6. $\begin{cases} R^2 - Rt - 5\,t^2 = 15, \\ R^2 - 6\,t^2 = 1. \end{cases}$

7. $\begin{cases} x^2 + xy - 2\,y^2 + 44 = 0, \\ xy + 3\,y^2 - 80 = 0. \end{cases}$

8. $\begin{cases} M^2 + MN + N^2 = 39, \\ 2\,M^2 + 3\,MN + N^2 = 63. \end{cases}$

9. $\begin{cases} a^2 - 2\,ab = -16, \\ 2\,ab + 4\,b^2 = 120. \end{cases}$

10. $\begin{cases} p^2 + pq + 2\,q^2 = 74, \\ 2\,p^2 + 2\,pq + q^2 = 73. \end{cases}$

11. $\begin{cases} F^2 - 2\,FC + 15 = 0, \\ C^2 - FC + F^2 = 21. \end{cases}$

12. $\begin{cases} g^2 - gv - 12\,v^2 = 8, \\ g^2 + gv - 10\,v^2 = 20. \end{cases}$

106. Symmetrical Equations. — A **symmetrical equation** is one that is unchanged when the unknown numbers are interchanged.

Thus, $m^2 + n^2 = 5$, $xy = 2$, and $a + b = 8$ are symmetrical.

Some systems involving only symmetrical equations, one linear and one quadratic, or both quadratic, may be solved by the methods of the foregoing sections. But another good method of solving such systems consists of so combining the equations as to find the values of the *sum* and the *difference* of the unknown numbers, and thus derive two or more systems of linear equations to which the given system is equivalent.

EXAMPLE. — Solve $\begin{cases} x^2 + y^2 = 40, & (1) \\ \quad\;\; xy = 12. & (2) \end{cases}$

Multiplying (2) by 2, $\quad 2\,xy = 24. \quad (3)$

Adding (3) to (1), $\quad x^2 + 2\,xy + y^2 = 64.$

Taking square root, $\quad x + y = \pm 8.$

Subtracting (3) from (1), $x^2 - 2\,xy + y^2 = 16$

Taking square root, $\quad x - y = \pm 4.$

Hence, the given system is equivalent to the four systems:

$$A\begin{cases} x+y=8, \\ x-y=4; \end{cases} \quad B\begin{cases} x+y=-8, \\ x-y=4; \end{cases} \quad C\begin{cases} x+y=8, \\ x-y=-4; \end{cases} \quad D\begin{cases} x+y=-8, \\ x-y=-4. \end{cases}$$

The solution of A is $x=6$, $y=2$; of B, $x=-2$, $y=-6$; of C, $x=2$, $y=6$; of D, $x=-6$, $y=-2$.

When, except for sign, the equations are symmetrical, the system may be solved by the same method.

EXAMPLE. — Solve $\begin{cases} m-n=2, & (1) \\ m^2+n^2=74. & (2) \end{cases}$

Squaring (1), $m^2-2mn+n^2=4.$ (3)

Subtracting (3) from (2), $2mn=70.$ (4)

Adding (4) and (2), $m^2+2mn+n^2=144.$

Taking square root, $m+n=\pm 12.$

Hence, the given system is equivalent to the two systems:

$$A\begin{cases} m-n=2, \\ m+n=12; \end{cases} \quad B\begin{cases} m-n=2, \\ m+n=-12. \end{cases}$$

The solution of A is $m=7$, $n=5$; of B, $m=-5$, $n=-7$.

NOTE. — In this and the preceding sections we have discussed methods of solving systems containing linear and quadratic equations of a few special types. Except in special cases such as those in the foregoing sections, to solve a system of two quadratic equations generally involves the solution of an equation of the *fourth degree* in one unknown number.

Thus, consider the system $\begin{cases} w+2v^2=7. & (1) \\ w^2+3v=6. & (2) \end{cases}$

Solving (1) for w, $w=7-2v^2.$

Substituting in (2), $49-28v^2+4v^4+3v=6,$

an equation of the fourth degree.

Consequently, we cannot undertake to solve all systems involving two quadratic equations.

EXERCISES

Solve:

1. $\begin{cases} x + y = 1, \\ x^2 + y^2 = 5. \end{cases}$

2. $\begin{cases} p + q = 3, \\ p^2 + q^2 = 29. \end{cases}$

3. $\begin{cases} a^2 + b^2 = 37, \\ ab = 6. \end{cases}$

4. $\begin{cases} m + n = 7, \\ mn = 6. \end{cases}$

5. $\begin{cases} P^2 + Q^2 = 13, \\ PQ = 6. \end{cases}$

6. $\begin{cases} w - v = 4, \\ w^2 + v^2 = 40. \end{cases}$

7. $\begin{cases} x - y = 8, \\ xy + 15 = 0. \end{cases}$

8. $\begin{cases} xy + 5 = 0, \\ x^2 + y^2 = 26. \end{cases}$

9. $\begin{cases} A^2 + AB + B^2 = 7, \\ A^2 + B^2 = 10. \end{cases}$

10. $\begin{cases} m^2 + mn + n^2 = 19, \\ m - n = 1. \end{cases}$

11. $\begin{cases} s + t = 4, \\ s^2 - st + t^2 = 7. \end{cases}$

12. $\begin{cases} x^2 + xy + y^2 = 4, \\ x^2 - xy + y^2 = 2. \end{cases}$

13. $\begin{cases} a + ab + b = 29, \\ a^2 + ab + b^2 = 61. \end{cases}$

14. $\begin{cases} W^2 - WV + V^2 = 63, \\ W - V + 3 = 0. \end{cases}$

15. $\begin{cases} r^2 + rr' + r'^2 = 31, \\ r^2 - rr' + r'^2 = 21. \end{cases}$

16. $\begin{cases} x^2 - 2xy + y^2 = 4, \\ x^2 + 2xy + y^2 = 64. \end{cases}$

107. Systems involving Higher Equations solved by Division. — We are able to solve some special systems involving equations of higher degree than the second. If the members of one equation are divisible by the members of the other, one or more simpler systems equivalent to the given system may be derived.

If A, B, M, and N represent integral expressions involving two unknown numbers, then the system

$$\begin{cases} MA = NB, & (1) \\ M = N & (2) \end{cases}$$

is equivalent to the two systems

$$(a) \begin{cases} A = B, & (3) \\ M = N, & (2) \end{cases} \quad \text{and} \quad (b) \begin{cases} M = 0, & (4) \\ N = 0. & (5) \end{cases}$$

For, by substituting N for M in (1), transposing, and factoring,

$$M(A-B)=0. \tag{6}$$

Also, from (2), $$M-N=0. \tag{7}$$

Now, the system composed of equations (6) and (7) is equivalent to the two systems (a and (b), because any solution of (a) or (b) will satisfy the equations (6) and (7), and any values of the unknown numbers satisfying (6) and (7) will satisfy the equations of (a) and (b).

Example. — Solve $\begin{cases} x^3+y^3=28x-7y-49, & (1) \\ x+y=4x-y-7. & (2) \end{cases}$

Dividing (1) by (2), $x^2-xy+y^2=7.$

Hence, the given system is equivalent to the two systems

$$A\begin{cases} x^2-xy+y^2=7, \\ x+y=4x-y-7, \end{cases} \text{ and } B\begin{cases} x+y=0, \\ 4x-y-7=0. \end{cases}$$

Solving A by the method of § 102,

$$x=1,\ y=-2, \text{ or } x=3,\ y=1.$$

Solving B, $x=1\tfrac{2}{5},\ y=-1\tfrac{2}{5}.$

If either M or N is an arithmetical number (not zero), the *system* (a) *alone* is equivalent to the given system, because in that case either equation (4) or (5) of system (b) would be impossible.

Example. — Solve $\begin{cases} v^3-t^3=56, & (1) \\ v^2+vt+t^2=28. & (2) \end{cases}$

Dividing (1) by (2), $v-t=2.$

Hence, the given system is equivalent to the single system

$$\begin{cases} v-t=2, \\ v^2+vt+t^2=28. \end{cases}$$

Solving this system by the method of § 102 or § 106,

$$v=4,\ t=2, \text{ or } v=-2,\ t=-4.$$

Observe that the system in the above example has only *two solutions*, whereas we expected *three solutions*. Such a system, in which the number of solutions is *less than the product of the degrees of the equations*, is called a **defective system**.

EXERCISES

Solve:

1. $\begin{cases} m^3+n^3=35, \\ m+n=5. \end{cases}$

2. $\begin{cases} v^3-t^3=27, \\ v-t=3. \end{cases}$

3. $\begin{cases} a^3-b^3=133, \\ a-b=7. \end{cases}$

4. $\begin{cases} x^3+y^3=65, \\ x+y=5. \end{cases}$

5. $\begin{cases} P^3+Q^3=28, \\ P+Q=4. \end{cases}$

6. $\begin{cases} a^2-ab+b^2=7, \\ a^3+b^3=28. \end{cases}$

7. $\begin{cases} w^3-v^3=56, \\ w^2+wv+v^2=28. \end{cases}$

8. $\begin{cases} M^3-N^3=19, \\ M^2+MN+N^2=19. \end{cases}$

9. $\begin{cases} x^4+x^2y^2+y^4=243, \\ x^2-xy+y^2=9. \end{cases}$

10. $\begin{cases} r^2+rs+s^2=13, \\ r^4+r^2s^2+s^4=91. \end{cases}$

11. $\begin{cases} a^2b+ab^2=30, \\ a+b=5. \end{cases}$

12. $\begin{cases} m^3-n^3=6\,m, \\ m-n=3\,m. \end{cases}$

13. $\begin{cases} 1-v^3=t^2, \\ 1-v=t. \end{cases}$

14. $\begin{cases} x^3+y^3=14\,x+21\,y-49, \\ x+y=2\,x+3\,y-7. \end{cases}$

15. $\begin{cases} m^3-n^3=62\,m-186\,n, \\ m-n=2\,m-6\,n. \end{cases}$

16. $\begin{cases} a^4+a^2+1=\frac{3}{7}\,b^2, \\ a^2+a+1=b. \end{cases}$

108. Other Systems involving Higher Equations. — Other systems involving equations of higher degree than the second may be solved by special devices.

EXAMPLE 1. — Solve

$$\begin{cases} x^4+y^4=641, & (1) \\ x+y=3. & (2) \end{cases}$$

Let $x=a+b$ and $y=a-b$.

Substituting in (2), $a+b+a-b=3.$

Hence, $$a=\tfrac{3}{2}. \tag{3}$$

Substituting in (1), $(a+b)^4+(a-b)^4=641,$

or
$$2\,a^4+12\,a^2b^2+2\,b^4=641. \tag{4}$$

From (3) and (4), $\frac{81}{8}+27\,b^2+2\,b^4=641,$

or
$$16\,b^4+216\,b^2-5047=0. \tag{5}$$

Solving (5) by quadratic formula, $b^2=\frac{49}{4}$ or $-\frac{103}{4}$.

Hence, $b=\pm\frac{7}{2}$ or $\pm\frac{1}{2}\sqrt{-103}$.

Using the value of a and each of the four values of b, we get

$$x=a+b=5,\ -2,\ \tfrac{3}{2}+\tfrac{1}{2}\sqrt{-103},\ \text{or}\ \tfrac{3}{2}-\tfrac{1}{2}\sqrt{-103}.$$

$$y=a-b=-2,\ 5,\ \tfrac{3}{2}-\tfrac{1}{2}\sqrt{-103},\ \text{or}\ \tfrac{3}{2}+\tfrac{1}{2}\sqrt{-103}.$$

EXAMPLE 2. — Solve $\begin{cases} a^4b^4+a^2b^2=20, & (1)\\ a+b=3. \end{cases}$

Factoring (1),
$$(a^2b^2+5)(a^2b^2-4)=0. \tag{2}$$

Hence,
$$a^2b^2+5=0, \tag{3}$$

or
$$a^2b^2-4=0. \tag{4}$$

Solving (3), $ab=\pm\sqrt{-5}$. Solving (4), $ab=\pm 2$.

Hence, the given system is equivalent to the four systems:

$A\begin{cases} a+b=3,\\ ab=\sqrt{-5}; \end{cases}$ $B\begin{cases} a+b=3,\\ ab=-\sqrt{-5}; \end{cases}$ $C\begin{cases} a+b=3,\\ ab=2; \end{cases}$ $D\begin{cases} a+b=3,\\ ab=-2. \end{cases}$

These systems will give in all eight solutions.

EXERCISES

Solve:

1. $\begin{cases} m^4+n^4=17,\\ m+n=3. \end{cases}$

2. $\begin{cases} w^4+v^4=257,\\ w-v=3. \end{cases}$

3. $\begin{cases} A^4+B^4=82,\\ A+B=4. \end{cases}$

4. $\begin{cases} x+y=5,\\ x^4+y^4=81. \end{cases}$

5. $\begin{cases} n+m=5,\\ n^4m^4-37\,n^2m^2+36=0. \end{cases}$

6. $\begin{cases} (a+b)^4-(a+b)^2=12,\\ a-b=0. \end{cases}$

7. $\begin{cases} h^6 - k^6 = 0, \\ h + 2k = 6. \end{cases}$

Suggestion. — By factoring find the linear and quadratic equations to which the first equation is equivalent. Use each of these with the second equation to form a new system.

8. $\begin{cases} x^6 - y^6 = 0, \\ x^2 + 2xy = 27. \end{cases}$

9. $\begin{cases} p^4 + p^2q^2 + q^4 = 0, \\ p^2 - 2pq + q^2 = 0. \end{cases}$

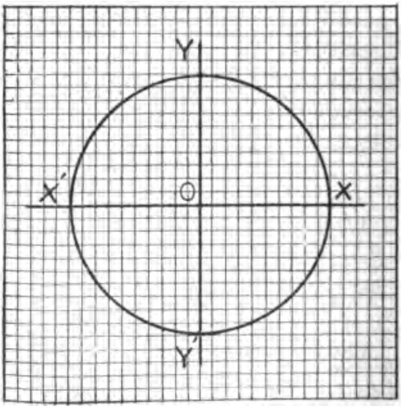
109. Graphs of Quadratic and Higher Equations. — The *graph* of an equation of the second or higher degree is in general some kind of *curve*. See First Course, § 153 and § 154.

Example 1. — Draw the graph of $x^2 + y^2 = 100$.

By assigning values to x and computing the corresponding values of y, the following sets of values are obtained :

x	0	2	4	6	8	10	-2	-4	-6	-8	-10
y	± 10	± 9.8	± 9.2	± 8	± 6	0	± 9.8	± 9.2	± 8	± 6	0

If x is given a value greater than $+10$ or less than -10, the corresponding value of y obtained is imaginary. This shows that the graph does not extend farther than 10 spaces to the right or left of the y-axis. Similarly, it does not extend farther than 10 spaces above or below the x-axis.



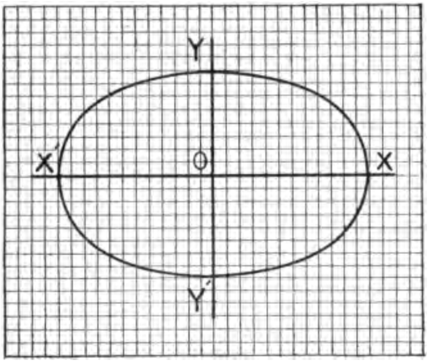
By locating the points whose coördinates are the above sets of values, and joining the consecutive points by a smooth curve, we get the circle in the figure.

Example 2. — Draw the graph of $4x^2 + 9y^2 = 576$.

By assigning values to x we get the following sets of values of x and y:

x	0	3	6	9	12	-3	-6	-9	-12
y	± 8	± 7.7	± 6.9	± 5.3	0	± 7.7	± 6.9	± 5.3	0

Marking the points whose coördinates are these sets of values, and connecting the consecutive points by a smooth line, gives the oval in the figure. This curve is called an **ellipse**.



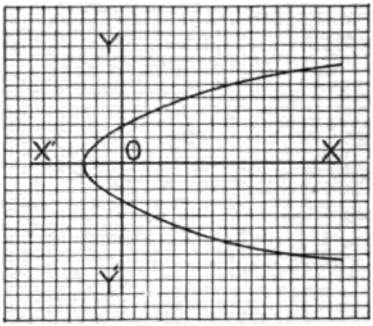
Example 3. — Draw the graph of $y^2 = 3x + 9$.

Proceeding as above, we get:

x	0	3	6	9	12	15	-3	Greater negative values
y	± 3	± 4.2	± 5.2	± 6	± 6.7	± 7.3	0	Imaginary

By locating the points whose coördinates are these sets of values, and connecting the consecutive points by a smooth line, we get the curve in the figure.

This curve is called a **parabola**. It is not a closed figure like the circle and ellipse, but its branches continue farther and farther apart as they recede from the y-axis.

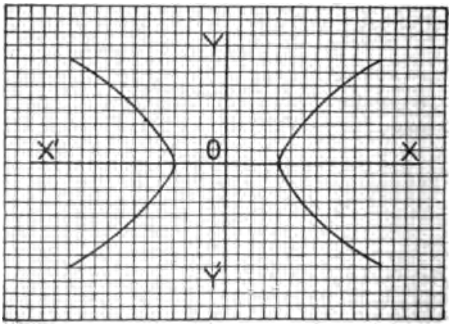


EXAMPLE 4. — Draw the graph of $x^2 - 2y^2 = 16$.

Proceeding as above, we get:

x	4	6	8	10	-4	-6	-8	-10
y	0	± 3.2	± 4.9	± 6.5	0	± 3.2	± 4.9	± 6.5

It is seen that for values of x between -4 and $+4$, y is imaginary. This shows that no part of the curve lies in the strip extending 4 spaces to the right and 4 spaces to the left of the y-axis.

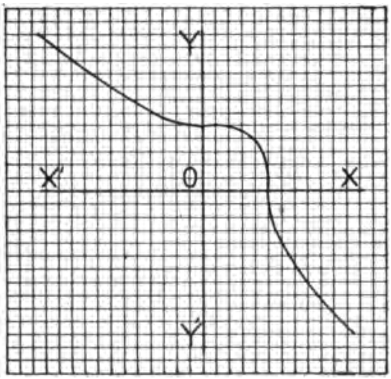


Drawing the curve, it is found to consist of two branches, as in the figure. This curve is called an **hyperbola**.

NOTE. — The curves obtained in the above examples play a very important part in our everyday life. Since the time of the early Greeks it has been known that they are the sections obtained by cutting a cone by a plane. The great astronomer Kepler (1571–1630) showed that the orbit in which the earth moves about the sun is an *ellipse*. It is now known that the orbits of all the planets are ellipses. Also, those comets that reappear periodically move in ellipses. Other comets that make their appearance only once move in *parabolas*. It is known also that all projectiles, such as an object thrown up in the air, or a shot fired from a cannon, move in parabolas. Thus, a knowledge of the parabola is used in gunnery in the navy, etc.

EXAMPLE 5. — Draw the graph of $x^3 + y^3 = 125$.

Proceeding as in the above examples, we get the curve shown in the figure.



It is seen that a straight line may be drawn cutting this curve in *three points*. Likewise, a straight line may be drawn cutting either of the above graphs of equations of the second degree in *two points*. Similarly, in general the graph of an equation of the fourth degree is such that it may be cut by a straight line in *four* points and only four, etc.

EXERCISES

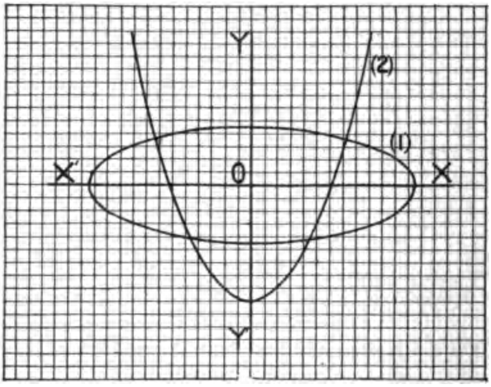
Draw the graphs of:

1. $x^2 + y^2 = 64$.
2. $x^2 + 4y^2 = 64$.
3. $9x^2 + 16y^2 = 576$.
4. $9x^2 - 4y^2 = 36$.
5. $x^2 = 2y$.
6. $xy = 8$.
7. $y^2 + 4x = 16$.
8. $x^3 + y^3 = 512$.
9. $x^3 - y^3 = 216$.

110. Systems solved Graphically. — A system involving two equations, of which one or both are of the second or higher degree, may be solved by drawing the graphs of the equations, and noting the coördinates of the points where the graphs cross. These coördinates must satisfy both equations, and hence form the solutions of the system. See § 53.

EXAMPLE. — Solve $\begin{cases} x^2 + 8y^2 = 160, & (1) \\ x^2 = 4y + 36. & (2) \end{cases}$

The graph of (1) is the ellipse (1), and the graph of (2) is the parabola (2). These curves meet at four points, whose coördinates are approximately $x = 7.1$, $y = 3.7$; $x = -7.1$, $y = 3.7$; $x = 4.4$, $y = -4.2$; $x = -4.4$, $y = -4.2$, the four solutions of the system.



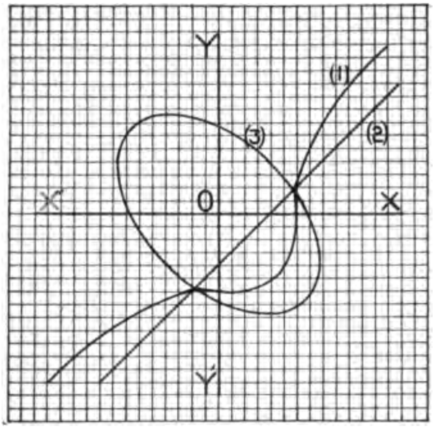
What takes place in solving a system when a new system equivalent to the given one is found and solved may be shown graphically.

EXAMPLE. — Interpret graphically the solution of $\begin{cases} x^3 - y^3 = 208, & (1) \\ x - y = 4. & (2) \end{cases}$

Dividing (1) by (2), $x^2 + xy + y^2 = 52.$ (3)

Now (2) and (3) form a new system whose solutions are found to be $x = 6$, $y = 2$ and $x = -2$, $y = -6$.

The graphs of (1), (2), and (3) are the curve (1), straight line (2), and ellipse (3) in the figure. It is seen that the only points where (1) and (2) cross are points where (2) and (3) cross, and conversely. Hence, to solve the system (1) and (2) is equivalent to solving the system (2) and (3), which has only the two solutions.



EXERCISES

Solve graphically:

1. $\begin{cases} x + 2y = 12, \\ x^2 + y^2 = 45. \end{cases}$

2. $\begin{cases} x^2 + y^2 = 100, \\ x - y = 5. \end{cases}$

3. $\begin{cases} 4x^2 + 9y^2 = 324, \\ y^2 = x + 16. \end{cases}$

4. $\begin{cases} x^2 - y^2 = 4, \\ y^2 - x = 12. \end{cases}$

5. $\begin{cases} x^2 + 4y^2 = 144, \\ 4x^2 + y^2 = 144. \end{cases}$

6. $\begin{cases} x^2 + y^2 = 81, \\ x^2 - y^2 = 16. \end{cases}$

7. $\begin{cases} x^2 - y^2 = 4, \\ x^2 - y = 16. \end{cases}$

8. $\begin{cases} x^2 + y^2 = 100, \\ x^2 + 2y^2 = 144. \end{cases}$

9. Show graphically that all solutions of the system

$$\begin{cases} x^2 + y^2 = 16, \\ x^2 - y^2 = 64, \end{cases}$$

are imaginary.

10. Show graphically that the following systems are equivalent

$$A\begin{cases} x^2-y^2=16, \\ x-y=2, \end{cases} \text{ and } B\begin{cases} x+y=8, \\ x-y=2. \end{cases}$$

11. Show graphically that the system

$$\begin{cases} x^2+y^2=40, \\ x+y=8, \end{cases}$$

is equivalent to the two systems

$$A\begin{cases} x+y=8, \\ x-y=4, \end{cases} \text{ and } B\begin{cases} x+y=8, \\ x-y=-4. \end{cases}$$

12. Show graphically that the system

$$\begin{cases} x^3+y^3=152, \\ x+y=8, \end{cases}$$

is defective.

111. Problems. — Some problems may be solved by use of systems of equations in two unknown numbers, in which one or both equations are of the second or higher degree.

EXERCISES

1. Find two numbers whose sum is 13 and the sum of whose squares is 97.

2. The area of a right triangle is 50 square feet. The hypotenuse is 15 feet. Find the legs.

3. A man has two fields, each square. It takes 40 rods more to fence one than the other. The larger contains $5\frac{5}{8}$ acres more than the smaller. Find the length of each.

4. The area of a rectangular floor is 1200 square feet. The diagonal of the floor is 50 feet. Find the dimensions.

5. The perimeter of a rectangle is 100 yards. Its area is 600 square yards. Find its dimensions.

6. The diagonal of a rectangle is 30 feet. If its length were increased 6 feet, its area would be increased 108 square feet. What are its dimensions?

7. The perimeter of a right triangle is 48 inches. Its area is 96 square inches. Find the three sides.

8. The floor of a room consists of a plain center surrounded by a parquetry border 3 feet wide. There are 140 square feet in the plain center and 180 square feet in the border. Find the dimensions of the room.

9. The area of a rectangle is 36 square inches. If its length be increased by 3 inches and its width by 2 inches, its area will be doubled. Find its dimensions.

10. A rectangle is 20 inches long and 16 inches wide. How much must be added to its width, and how much must be taken from its length, in order that its area may be increased by 22 square inches and its perimeter by 2 inches?

11. The hypotenuse of a right triangle is 10. If one leg be increased by 3 and the other leg by 4, the hypotenuse will become 15. Find the sides of the triangle.

12. A rectangular lot which contains 880 square yards is 2.2 times as long as it is wide. How much will it cost to build a fence around it at 25 cents a linear foot?

13. A guy rope to a derrick is attached to a stake 30 feet from the foot of the derrick. If the rope were $16\frac{2}{3}$ feet longer, it would reach to a stake $53\frac{1}{3}$ feet from the foot of the derrick. Find the height of the derrick and the length of the rope.

14. A flower garden contains 1000 square feet, and is surrounded by a path 5 feet wide. The area of the path is 750 square feet. What are the dimensions of the garden?

15. A laborer received $32.50 wages. If he had worked 5 days longer, and had received 50 cents a day less, he would have received \$41.25. How long did he work, and what were his wages a day?

16. A certain number of men do a piece of work in a certain number of days. If there were two fewer men, it would take 4 days longer to do the work; and if there were twice as many men, it would take 4 days less to do the work. Find the number of men and the number of days required for them to do the work.

17. A grocer bought apples and potatoes for \$54. He sold the apples for \$36.80 and the potatoes for \$18.70. He gained as many per cent on the apples as he lost on the potatoes. How much did he pay for each?

18. A certain sum of money placed at simple interest for one year amounted to \$265. If the principal had been \$50 more and the rate 1% less, the interest would have been the same. Find the principal and the rate.

19. The sides of a triangle are 10, 12, and 14. Find the altitude on the side 14.

Suggestion. — The altitude divides the given triangle into two right triangles. Call the base of one of these triangles x. Then the other side will be $14 - x$. Draw the figure.

20. The sides of a triangle are 12, 16, and 20. Find the altitude on side 20, and also find the area of the triangle.

21. One season an agent sold a lot of reapers at a uniform price, receiving \$4800 for all. Another agent sold one more reaper, but received \$8 less for each. They found that each had received the same amount for all. Find the number sold by each, and the price of each.

22. A and B start simultaneously from towns 247 miles apart and travel towards each other until they meet. A travels 9 miles per day. The number of days before they meet is 3 greater than the number of miles B travels per day. How far does each travel?

23. A and B both start from the same place and travel in the same direction. But A starts 2 hours before B, and is overtaken by him 30 miles from the starting point. Had each traveled half a mile more per hour, B would have overtaken A just 42 miles from the starting point. Find the rate of travel of each.

24. A rectangular field contains 1536 square rods. The length and width of another field are each just 5 rods less, and its area is 375 square rods less. Find the dimensions of the first field.

25. The sum of two numbers is 60. The sum of their squares divided by their product is 2.5. Find the numbers.

26. The fore wheel of a carriage makes 6 revolutions more than the hind wheel in going 120 yards. But if the circumference of each wheel is increased by 3 feet, the fore wheel will make but 4 more revolutions than the hind wheel in going 120 yards. Find the circumference of each.

SUPPLEMENTARY EXERCISES

Solve:

1. $\begin{cases} m^2 + n^2 - m - n = 22, \\ m + n + mn = -1. \end{cases}$

Suggestion. — Multiply members of second equation by 2, and add to first.

2. $\begin{cases} x^2 - y^2 = x + y, \\ x^2 - 3\,xy = 5\,x - 15\,y. \end{cases}$

Suggestion. — By factoring, find linear equations equivalent to each equation.

3. $\begin{cases} A^4 + A^2B^2 + B^4 = 91, \\ A^2 + AB + B^2 = 13. \end{cases}$

4. $\begin{cases} p^2q - p = 50, \\ p^4q^2 - p^2 = 2475. \end{cases}$

5. $\begin{cases} x^2 + y^2 + 7\,xy = 171, \\ xy = 2(x + y). \end{cases}$

6. $\begin{cases} r^2 + s^2 - (r - s) = 20, \\ rs + r - s = 1. \end{cases}$

7. $\begin{cases} M^2N + MN^2 = 180, \\ M^2N^2 = 400. \end{cases}$

8. $\begin{cases} (x + y)(x - 4\,y) = 4\,y - x, \\ x^2 - 2\,xy + y^2 = 3(x - y). \end{cases}$

9. $\begin{cases} w^2v^2 + wv = 6, \\ w^2 + v^2 = 5. \end{cases}$

10. $\begin{cases} d^4 = 9\,t^4 + 48, \\ d^2 = 3\,t^2 + 2. \end{cases}$

11. $\begin{cases} P^2 + Q = 7, \\ Q^2 + P = 11 \end{cases}$

12. $\begin{cases} a^2 + ab + b = 0, \\ a^2 + ab + a = 0. \end{cases}$

13. $\begin{cases} h^2 = 8h + 6k, \\ k^2 = 6h + 8k. \end{cases}$

14. $\begin{cases} \sqrt{x} + \sqrt{y} = 8, \\ x^2 + y^2 = 706. \end{cases}$

Suggestion. — Let

$$\sqrt{x} = v, \quad \sqrt{y} = w.$$

15. $\begin{cases} m - n = 19, \\ \sqrt[3]{m} - \sqrt[3]{n} = 1. \end{cases}$

Suggestion. — Let

$$\sqrt[3]{m} = v, \quad \sqrt[3]{n} = w.$$

16. $\begin{cases} \sqrt{a} + \sqrt{b} = 2, \\ \sqrt{a^3} + \sqrt{b^3} = 26. \end{cases}$

17. $\begin{cases} x + y + 2\sqrt{x + y} = 24, \\ x - y + 3\sqrt{x - y} = 10. \end{cases}$

18. $\begin{cases} w + \sqrt{wv} + v = 14, \\ w^2 + wv + v^2 = 84. \end{cases}$

19. $\begin{cases} \dfrac{1}{x} - \dfrac{1}{y} = 1, \\ \dfrac{1}{x^3} - \dfrac{1}{y^3} = 19 \end{cases}$

CHAPTER XI

PROPORTION. VARIABLES

112. Ratio. — The **ratio** of two numbers of the same kind is the quotient of the first divided by the second. It is usually written in the form of a fraction.

Thus, the ratio of 12 feet to 20 feet is written $\frac{12 \text{ feet}}{20 \text{ feet}}$. The ratio of m to n is written $\frac{m}{n}$. An old form of writing the ratio of m to n, now little used, is $m : n$.

All rules applicable to fractions apply also to ratios, such as the rule for reduction to lower terms, etc.

The dividend or numerator of a ratio is sometimes called the **antecedent**, and the divisor or denominator the **consequent**.

Two quantities are called **commensurable** when their ratio can be expressed by two whole numbers, *i.e.* when a unit of measure can be found that is contained an integral number of times in each. Two quantities that are not commensurable are called **incommensurable.**

EXERCISES

1. Express the ratio of 6 pounds to 8 pounds as a fraction in its lowest terms.

2. The tax rate in a certain community is \$3 to every \$100 of property. Express this rate or ratio as a common fraction. As a decimal.

3. The death rate in the United States in a recent year, due to street car accidents, was given as 4 to 100,000 population. Express this rate or ratio as a common fraction in its lowest terms. As a decimal.

4. What is the ratio of the circumference to the diameter of a circle? Are the circumference and the diameter commensurable? Why not?

5. One cubic foot of iron weighs 450 pounds, and one cubic foot of water weighs 62.5 pounds. Express the ratio of the weight of iron to the weight of an equal volume of water as a common fraction in its lowest terms. As a decimal.

6. Express the ratio of $m+n$ to m^3+n^3 as a fraction in its lowest terms.

7. Express the ratio of $\frac{2}{3}$ to $\frac{5}{6}$ as a simple fraction.

8. Express the ratio of $\frac{v}{t}$ to $\frac{t'}{v'}$ as a simple fraction.

9. Show that $\frac{7}{8}$ and $\frac{9}{10}$ are commensurable.

10. Show that any two fractions whose terms are whole numbers are commensurable.

11. Show that 4 and $\sqrt{2}$ are not commensurable.

12. Are $\sqrt{32}$ and $\sqrt{8}$ commensurable?

13. Show that the diagonal and side of a square are not commensurable.

14. Which ratio is the greater, $\frac{5}{8}$ or $\frac{29}{47}$?

15. Divide a profit of $2400 between two partners so that their shares shall be in the ratio of 2 to 3.

113. Proportion. — An equation each of whose members consists of a ratio is called a **proportion**.

Thus, $\frac{3\text{ lb.}}{8\text{ lb.}}=\frac{6\text{ mi.}}{16\text{ mi.}}$, $\frac{2m}{5n}=\frac{6m}{15n}$, and $\frac{a}{b}=\frac{c}{d}$ are proportions.

The proportion $\frac{a}{b}=\frac{c}{d}$ is read "a is to b as c is to d," as well as "a divided by b equals c divided by d."

In any proportion $\frac{a}{b}=\frac{c}{d}$, the numbers a, b, c, and d are called the **terms**. The first and fourth terms, a and d, are called the

extremes. The second and third terms, b and c, are called the **means.**

The four terms of a proportion are sometimes said to be "proportional," or "in proportion."

If $\frac{a}{b}=\frac{c}{d}$, the fourth term d is called the **fourth proportional** to a, b, and c.

A proportion such as $\frac{a}{n}=\frac{n}{b}$, in which the means are equal, is called a **mean proportion.** The mean term n is called a **mean proportional** between a and b. And the fourth term b is called the **third proportional** to a and n.

Since a proportion is an equation, all processes which may be performed upon an equation may be performed upon a proportion.

Since the ratio of any two concrete numbers is the same as that of the corresponding abstract numbers, in forming a proportion confusion may be avoided by writing only the abstract numbers.

Thus, the ratio $\frac{2 \text{ oz.}}{7 \text{ oz.}}$ may be expressed by $\frac{2}{7}$.

EXERCISES

Find the value of the unknown term in each of the following proportions:

1. $\frac{x}{5}=\frac{9}{15}$.
2. $\frac{t}{8}=\frac{1}{12}$.
3. $\frac{6}{v}=\frac{16}{9}$.
4. $\frac{12}{20}=\frac{8}{r}$.
5. $\frac{7}{9}=\frac{P}{27}$.
6. $\frac{8}{21}=\frac{4}{k}$.
7. $\frac{1}{y}=\frac{10}{36}$.
8. $\frac{84}{32}=\frac{F}{42}$.

Find the value of n in each of the proportions:

9. $\frac{x}{y}=\frac{n}{z}$.
10. $\frac{v}{2n}=\frac{t}{s}$.
11. $\frac{1}{k}=\frac{k}{n}$.
12. $\frac{a}{b}=\frac{cn}{a}$.
13. $\frac{r-n}{r}=\frac{d+n}{d}$.
14. $\frac{2\pi r}{n}=\frac{2\pi r'}{n+r}$.

Find the fourth proportional to:

15. 3, 4, and 6.

16. 9, 2, and 12.

17. 1, 20, and $\frac{1}{2}$.

18. $\frac{1}{2}$, $\frac{1}{4}$, and $\frac{1}{8}$.

19. 1, x^2, and $2x$.

20. 1, $m+n$, and $m-n$.

Find the third proportional to:

21. 3 and 12.

22. 8 and 1.

23. $\frac{3}{4}$ and $\frac{1}{6}$.

24. w and wv.

25. p and q.

26. $a-b$ and a^2-b^2.

Find the mean proportional between:

27. 1 and 16.

28. 9 and 25.

29. 6 and 54.

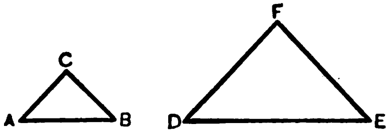
30. 5 and 7.

31. a and b.

32. V and $V-2V^2+V^3$.

33. What number must be added to each of the four numbers 1, 17, 5, 29 to make the results proportional?

34. In a city whose population was 96,000 the number of Germans was 12,000. When the total population increased to 120,000, the number of Germans increased proportionally. Of the 120,000 population how many were Germans?



35. It is known in geometry that in two similar triangles the corresponding sides are proportional; that is,

$$\frac{AB}{DE}=\frac{AC}{DF}.$$

If $AB=3$ inches, $AC=2$ inches, and $DE=7$ inches, find the length of DF.

36. In two similar triangles the sides of the smaller are 8 inches, 10 inches, and 12 inches, and the longest side of the larger is 40 inches. Find the other sides of the larger triangle.

37. A tree casts a shadow 60 feet long when a post 4 feet high casts a shadow 5 feet 8 inches long. How high is the tree?

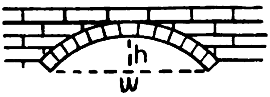
38. The line which divides an angle of a triangle into equal parts divides the opposite side into two parts which are proportional to the adjacent sides. The sides of a triangle are 12 inches, 16 inches, and 18 inches. The angle formed by the 12-inch and 16-inch sides is divided by a line into two equal parts. Find the lengths of the parts into which it divides the 18-inch side.

39. The sides of a triangle are 10 inches, 14 inches, and 15 inches. Find the lengths of the parts into which the 14-inch side is divided by the line which divides the opposite angle into equal parts.

40. The problem to divide a given line into two parts such that the longer part shall be a mean proportional between the whole line and the shorter part has been called since the time of the ancient Greeks the Problem of the Golden Section. A line is 16 inches long. Find the lengths of the parts into which it is divided by the golden section.

41. If a line is 18 feet long, find the lengths of the parts into which it is divided by the golden section.

42. If the radius of a circular arch is r, the width or span w, and the height or rise h, as shown in the figure, then the half-span is a mean proportional between the rise and the difference between the diameter and rise; that is,



$$\frac{h}{\frac{1}{2}w}=\frac{\frac{1}{2}w}{2r-h}.$$

If the stone arch of a doorway is to have a span of 10 feet and rise of 4 feet, find the radius of the circle to which the stones of the arch must be cut.

43. The span of a bridge is 80 feet and the rise of the arch 16 feet. Find the radius of the arch.

44. A lever is a stiff bar by means of which a great resistance may be overcome through the application of a small force or effort. The bar revolves on a pivot called the fulcrum, the weight or resistance is applied at one end, and the effort at the other.

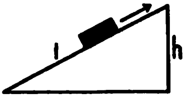
If the resistance is W, its distance from the fulcrum D, the effort w, and its distance from the fulcrum d, then

$$\frac{w}{W}=\frac{D}{d}.$$

If $W=1200$ pounds, $D=4$ inches, and $d=60$ inches, find w.

45. What effort is required to lift a stone weighing 720 pounds by means of a crowbar whose entire length is 54 inches, if the fulcrum is placed 3 inches from the stone.

46. If an object is moved up an inclined plane or slope, neglecting friction, the force applied parallel to the incline required to move the object is to the weight of the object as the height of the incline is to its length. That is, if W is the weight of the object, F the force required to move it, h the height of the incline, and l the length of the incline, then



$$\frac{F}{W}=\frac{h}{l}.$$

If $W=1200$ pounds, $l=40$ feet, and $h=8$ feet, find F.

47. What force is required to roll a barrel weighing 216 pounds up a board and into a wagon, if the board is 10 feet long, and the wagon bed 30 inches from the ground?

48. What force is required to move a block of ice weighing 200 pounds up an incline 80 feet long and 36 feet high?

114. General Principles in Proportion. — The following principles, which are often used in the applications of proportion, were nearly all established in the FIRST COURSE, § 119. Let the student give the proof of each of them.

(1) *If four numbers are in proportion, the product of the extremes equals the product of the means.* That is,

$$\text{if } \frac{a}{b}=\frac{c}{d}, \text{ then } ad=bc.$$

(2) *If the product of two numbers equals the product of two other numbers, the four numbers are in proportion.* That is,

$$\text{if } ad=bc, \text{ then } \frac{a}{b}=\frac{c}{d}.$$

(3) *If four numbers are in proportion, they are in proportion by inversion.* That is,

$$\text{if } \frac{a}{b}=\frac{c}{d}, \text{ then } \frac{b}{a}=\frac{d}{c}.$$

(4) *In any proportion, the means may be interchanged, or the extremes interchanged, without destroying the proportion.* That is,

$$\text{if } \frac{a}{b}=\frac{c}{d}, \text{ then } \frac{a}{c}=\frac{b}{d} \text{ and } \frac{d}{b}=\frac{c}{a}.$$

(5) *The terms of any proportion are in proportion by addition.* That is,

$$\text{if } \frac{a}{b}=\frac{c}{d}, \text{ then } \frac{a+b}{b}=\frac{c+d}{d}.$$

(6) *The terms of any proportion are in proportion by subtraction.* That is,

$$\text{if } \frac{a}{b}=\frac{c}{d}, \text{ then } \frac{a-b}{b}=\frac{c-d}{d}.$$

(7) *Like powers of the terms of a proportion are in proportion.* That is,

$$\text{if } \frac{a}{b}=\frac{c}{d}, \text{ then } \frac{a^2}{b^2}=\frac{c^2}{d^2}, \frac{a^3}{b^3}=\frac{c^3}{d^3}, \text{ etc.}$$

(8) *Like principal roots of the terms of a proportion are in proportion.* That is,

$$\text{if } \frac{a}{b}=\frac{c}{d}, \text{ then } \frac{\sqrt{a}}{\sqrt{b}}=\frac{\sqrt{c}}{\sqrt{d}},\ \frac{\sqrt[3]{a}}{\sqrt[3]{b}}=\frac{\sqrt[3]{c}}{\sqrt[3]{d}}, \text{ etc.}$$

(9) *If two or more ratios are equal, the sum of the antecedents is to the sum of the consequents as any antecedent is to its consequent.* That is,

$$\text{if } \frac{a}{b}=\frac{c}{d}=\frac{e}{f}=\text{etc., then } \frac{a+c+e+\text{etc.}}{b+d+f+\text{etc.}}=\frac{a}{b}=\text{etc.}$$

Suggestion. — Let each of the given ratios equal r. Thus, $\frac{a}{b}=r$, etc. Clear the equations obtained of fractions, add, factor, etc.

By use of the above principles new proportions may be derived from any given proportion.

Example. — If $\frac{a}{b}=\frac{c}{d}$, show that $\frac{\sqrt{a^2+b^2}}{\sqrt{c^2+d^2}}=\frac{\sqrt[3]{a^3-b^3}}{\sqrt[3]{c^3-d^3}}$.

By (7), $$\frac{a^2}{b^2}=\frac{c^2}{d^2}.$$

By (5), $$\frac{a^2+b^2}{b^2}=\frac{c^2+d^2}{d^2}.$$

By (4), $$\frac{a^2+b^2}{c^2+d^2}=\frac{b^2}{d^2}.$$

By (8), $$\frac{\sqrt{a^2+b^2}}{\sqrt{c^2+d^2}}=\frac{b}{d}.$$

Similarly, by applying in succession (7), (6), (4), and (8) to $\frac{a}{b}=\frac{c}{d}$,

$$\frac{\sqrt[3]{a^3-b^3}}{\sqrt[3]{c^3-d^3}}=\frac{b}{d}.$$

Hence, $$\frac{\sqrt{a^2+b^2}}{\sqrt{c^2+d^2}}=\frac{\sqrt[3]{a^3-b^3}}{\sqrt[3]{c^3-d^3}}, \text{ since each equals } \frac{b}{d}.$$

EXERCISES

If $\frac{a}{b}=\frac{c}{d}$, show that:

1. $\frac{a}{a+b}=\frac{c}{c+d}$.

2. $\frac{a^2-b^2}{a^2}=\frac{c^2-d^2}{c^2}$.

3. $\frac{a+b}{c+d}=\frac{\sqrt{a^2+b^2}}{\sqrt{c^2+d^2}}$.

4. $\frac{a-b}{c-d}=\frac{\sqrt[3]{a^3-b^3}}{\sqrt[3]{c^3-d^3}}$.

5. $\frac{a-2b}{b}=\frac{c-2d}{d}$.

6. $\frac{2a^2+3b^2}{b^2}=\frac{2c^2+3d^2}{d^2}$.

7. $\frac{na-b}{nc-d}=\frac{a}{c}$.

8. $\frac{a+kb}{a-kb}=\frac{c+kd}{c-kd}$.

9. $\frac{a^2+b^2}{2ab}=\frac{c^2+d^2}{2cd}$.

10. $\frac{a^4-4b^4}{b^4}=\frac{c^4-4d^4}{d^4}$.

11. $\frac{a^3+b^3}{3a^2b+3ab^2}=\frac{c^3+d^3}{3c^2d+3cd^2}$.

12. $\frac{a}{b}=\frac{\sqrt{ac}}{\sqrt{bd}}$.

13. $\frac{a^2c+ac^2}{b^2d+bd^2}=\frac{(a+c)^3}{(b+d)^3}$.

14. $\frac{a}{a+c}=\frac{a+b}{a+b+c+d}$.

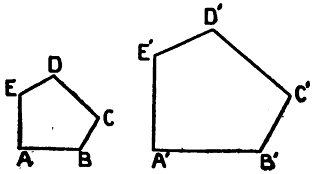
15. If $\frac{a}{b}=\frac{c}{d}=\frac{e}{f}$, show that $\frac{a^2+c^2+e^2}{b^2+d^2+f^2}=\frac{c^2}{d^2}$.

16. If $\frac{a}{b}=\frac{c}{d}=\frac{e}{f}$, show that $\frac{a-c+e}{b-d+f}=\frac{a}{b}$.

17. If $\frac{a}{b}=\frac{c}{d}=\frac{e}{f}$, show that $\frac{a^2+2c^2+3e^2}{b^2+2d^2+3f^2}=\frac{a^2}{b^2}$.

18. In the similar polygons shown in the figure it is known that

$$\frac{AB}{A'B'}=\frac{BC}{B'C'}=\frac{CD}{C'D'}=\text{etc.}$$



Show that the perimeters are to each other as any two corresponding sides; that is,

$$\frac{AB + BC + CD + \text{etc.}}{A'B' + B'C' + C'D' + \text{etc.}} = \frac{AB}{A'B'}.$$

115. Variables. — Any quantity which is continuously changing is called a **variable.** Any quantity which does not change is called a **constant.**

Thus, the temperature at a given place, a man's age, the cost of living, etc., are variables.

Some variables are so related to each other that the values of one depend upon the values of the others.

The relation between such variables may be expressed by an equation.

Thus if an object is let fall toward the ground, the velocity at which it falls gradually increases until the object strikes the ground. If the object falls from a great height, its velocity at the end of 1 second is 32 feet per second; at the end of 2 seconds it is 64 feet per second; at the end of 3 seconds it is 96 feet per second; etc.

Evidently, the velocity that the object may have at any point of its fall depends upon the length of time that it has been falling. If v represents the velocity in feet per second when the object has been falling for t seconds,

then $$v = 32\,t.$$

NOTE. — The above law holds for an object falling in a vacuum. If an object falls through the open air, the resistance of the air tends to retard the fall.

116. Direct Variation. — If two variables so change that their *ratio is constant*, one is said to **vary directly as the other.** As one increases the other also increases, and as one decreases the other also decreases.

If x and y are the variables, then $\frac{x}{y} = k$, where k is some constant.

The fact that x varies directly as y is sometimes expressed also by the symbol $\propto$, thus:

$$x \propto y.$$

It is evident that when one quantity varies directly as another the ratio of any two corresponding values of the variables equals the ratio of any other two corresponding values, since each ratio equals the same constant.

Thus, if x varies directly as y, and x' and y', and x'' and y'' are any two pairs of corresponding values of x and y, then

$$\frac{x'}{y'}=\frac{x''}{y''}.$$

Hence, if one quantity varies directly as another, any two pairs of corresponding values of the quantities form a proportion.

EXAMPLE. — The weight of an object on or below the surface of the earth varies directly as its distance from the center of the earth. If an object weighs 120 pounds at the surface of the earth, find its weight 1000 miles below the surface.

Let w be the required weight in pounds.

Then $$\frac{w}{120}=\frac{3000}{4000}.\text{*}$$

Solving, $$w=90.$$

117. Inverse Variation. — One variable is said to **vary inversely** as another when, during all of their changes, their *product remains constant.* As one increases, the other must decrease.

If x and y are the variables, then $xy=k$, where k is some constant.

It is evident that the product of any two corresponding values of the variables equals the product of any other two corresponding values, since each product equals the same constant.

Thus, if x varies inversely as y, and x' and y' and x'' and y'' are any two pairs of corresponding values, then

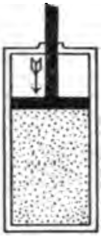
$$x'y'=x''y''.$$

This principle may be applied as in the following example.

---

* 4000 miles is taken to be the radius of the earth.

Example. — It is a law of gases that the volume of a given quantity of a gas varies inversely as the pressure upon it. Gas in a cylinder occupies 60 cubic inches when the pressure upon it is 8 pounds. What will be the volume if the pressure is increased to 42 pounds?



Let V be the required volume, in cubic inches.

Then $\qquad 42V = 8 \times 60.$

Solving, $\qquad V = 11\frac{3}{7}.$

118 Other Forms of Variation. — One variable is said to vary **jointly** as two or more other variables when it varies as their product.

If x varies jointly as y and z, then $\frac{x}{yz} = k$, or $x = kyz$, where k is some constant.

Thus, the area of a triangle varies jointly as its altitude and base. If A = area, h = altitude, and b = base, then $A = \frac{1}{2}hb$.

One variable is said to **vary directly as a second and inversely as a third** when it varies directly as the quotient of the second divided by the third.

One variable may also vary directly or inversely as the square, cube, square root, etc., of another. In general, one variable may vary as any algebraic expression, involving one or more other variables.

Thus, $V = \pi R^2 H$ is the formula for computing the volume of a cylinder. It shows that the volume varies jointly as the altitude and the square of the radius.

Two quantities x and y may so vary that $y = 2x^2 - 3x + 8$, or $y = x^3 - 4$, etc. For any value that x takes, y assumes a corresponding value.

EXERCISES

1. Write $v \propto t$ as an equation; also express this variation of v and t by a proportion between two sets of corresponding values of v and t.

2. In a bicycle pump the volume v of air confined varies inversely as the pressure p upon the piston. Express the relation between v and p in three ways.

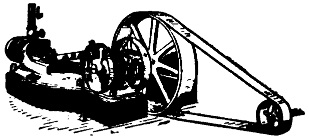
3. If n varies directly as d, and $n = 10$ when $d = 2$, find n when $d = 9$.

4. If R varies inversely as t, and $R = \frac{1}{2}$ when $t = 8$, find R when $t = 60$.

5. If W varies directly as F and inversely as d, and $W = 120$ when $F = 8$ and $d = 36$, find W when $F = 20$ and $d = 12$.

6. When air is pumped into an automobile tire, the pressure required varies inversely as the volume. If the pressure is 40 pounds when the volume is 140 cubic inches, what is the pressure when the volume is 100 cubic inches?

7. The number of revolutions per minute made by a pulley driven by a belt, as in the figure, varies inversely as the diameter of the pulley. If a pulley which is 24 inches in diameter and makes 60 revolutions per minute were replaced by a pulley 16 inches in diameter, how many revolutions per minute would the latter make, if the belt runs at the same speed?



8. If an engine driving a 12-inch pulley gives it 120 revolutions per minute, how many revolutions per minute would it cause a 20-inch pulley to make when running at the same speed?

9. If an object weighs 180 pounds at the surface of the earth, what would be its weight 500 miles below the surface of the earth? 2000 miles? 3000 miles? See Example, § 116.

10. The weight of an object above the surface of the earth varies inversely as the square of its distance from the center of the earth. If an object weighs 150 pounds at the surface of the earth, what would it weigh 1000 miles above the surface of the earth? 4000 miles? 10,000 miles?

11. The distance in feet that an object will fall from rest varies directly as the square of the time in seconds. A body will fall 16 feet the first second. How far will it fall in 4 seconds? In 10 seconds? In 60 seconds?

12. The pressure of the wind against a flat surface, such as a wall, varies jointly as the area of the surface and the square of the velocity of the wind in miles per hour. The pressure on one square foot when the velocity is 20 miles per hour is 2 pounds. Find the pressure on a wall 10 feet by 30 feet when the velocity is 40 miles per hour.

13. The side of a house is 48 feet long and 20 feet high. Find the pressure against it when the wind is blowing 30 miles per hour.

14. The intensity or brightness of light upon a surface varies inversely as the square of the distance of the surface from the source of the light. If the intensity of light on a surface from a lamp 4 feet away is 8 candle power, what would be the intensity if the lamp were removed to a distance of 16 feet from the surface?

15. An oil lamp and a 16 candle power electric light illuminate a surface with equal intensity where the oil lamp is 4 feet and the electric light 8 feet from the surface. Find the candle power of the lamp.

16. Three standard candles one foot from a wall and an arc light 5 feet from the wall illuminate it equally. Find the candle power of the arc light.

17. A light shines on the page of a book 4 feet away. How far from the light must the book be held to receive 4 times as much light?

18. The time required for the pendulum of a clock to make a vibration varies directly as the square root of its length. A pendulum 100 centimeters long makes a vibration in 1 second. Find the time of vibration of a pendulum 81 centimeters long. Of one 64 centimeters long. Of one 16 centimeters long.

19. Find the length of a pendulum which vibrates in 2 seconds. In 4 seconds. In 10 seconds.

20. The amount of water delivered by a pipe per minute varies directly as the square of the diameter of the pipe. If a 3-inch pipe delivers 200 gallons per minute, how much would a 4-inch pipe deliver?

21. The squares of the times of revolution of the planets about the sun vary directly as the cubes of their distances from the sun. The earth makes a revolution about the sun in approximately 365 days, and is 93,000,000 miles from the sun. Mercury is 36,000,000 miles from the sun. How long does it take Mercury to make a revolution about the sun?

22. Jupiter is 483,300,000 miles from the sun. How long does it take Jupiter to make a revolution about the sun?

119. Variation shown Graphically. — Variation of two quantities may be depicted graphically to the eye by expressing the relation between the variables by an equation, and drawing the graph of the equation. The values of one variable corresponding to values assumed by the other may then be determined at sight from the graph.

EXAMPLE. — The volume of a gas varies inversely as the pressure upon it. (See Example, § 117.) When the pressure is 5 pounds the volume is 2 cubic feet. Represent graphically the variation of the volume and pressure.

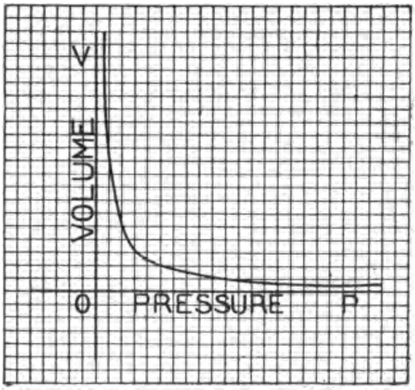
If the volume is denoted by V and the pressure by P,

$$VP = 10.$$

Since, from the nature of the quantities, only positive values of V and P may be used, the graph is found to be the branch of an hyperbola shown in the figure.

This curve shows that as P increases V decreases, that is, the curve approaches nearer and nearer to the P-axis. And as P decreases V increases.

The value of V corresponding to any value of P may be seen at a glance from the curve. Thus, when $P = 20$, $V = \frac{1}{2}$.



EXERCISES

1. Show that the fact that x varies directly as y is represented graphically by a straight line.

2. If x varies directly as y, and $x = 4$ when $y = 2$, draw the graph showing the variation of x and y.

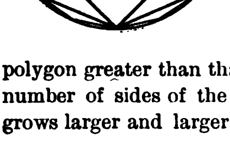
3. From the graph in Ex. 2, tell the value of x when y is 20; 40; 45; 18.

4. The distance a body falls from rest varies directly as the square of the time of falling. A body falls 16 feet the first second. Draw the graph showing the variation of the distance and time.

5. An object is thrown upward at an angle of 45° with horizontal, and with such a force that its horizontal velocity is 16 feet per second. If x represents the distance that the object moves horizontally in any time and y the distance that it moves vertically in the same time, then $y = x - \frac{1}{16}x^2$. By drawing the graph of this equation, determine the path of the object.

6. Draw the graph showing the variation of the length and the time of vibration of a pendulum. See Ex. 18, § 118.

120. The Limit of a Variable. — If the value of a variable changes according to some law, taking on a series of values that approach nearer and nearer to some constant value in such a manner that the difference between the variable and the constant *becomes and remains* less than any assigned value, however small, then the constant is called the **limit of the variable.**



Thus, if a square is drawn with its corners or vertices in a circle; then this square replaced by a polygon with eight equal sides, then this polygon replaced by another with sixteen equal sides, etc., the area of the eight-sided polygon is greater than that of the square, the area of the sixteen-sided polygon greater than that of the eight-sided polygon, etc. Evidently, as the number of sides of the polygon is thus increased, the area of the polygon grows larger and larger, and if the number of sides is indefinitely increased, the difference between the area of the polygon and that of the circle becomes and remains less than any fixed quantity, however small. Hence, as the number of sides is indefinitly increased, the *area of the polygon approaches the area of the circle as a limit.*

The sign $\doteq$ placed between a variable and a constant indicates that the variable approaches the constant as a limit.

Thus, $x \doteq a$ means that x approaches a as a limit.

EXERCISES

1. What is the limit approached by the perimeter of the polygon in § 120 as the number of sides of the polygon is increased indefinitely?

2. What is the limit of the sum of the fractions $\frac{1}{2}$, $\frac{1}{4}$, $\frac{1}{8}$, $\frac{1}{16}$, $\frac{1}{32}$, etc., as the number of fractions taken is increased indefinitely?

SUGGESTION. — Find the sum of the first two fractions; then the sum of the first three; etc. See what number the sum approaches but never quite equals, as more and more fractions are taken.

3. What is the limit of 0.3333 ⋯ as the number of decimal places taken is increased indefinitely?

4. If $n \doteq 0$, what limit does $2n$ approach? n^2? $n + 6$?

5. If $t \doteq 2$, what limit does $t^2 - 4$ approach?

6. If x increases indefinitely, what limit does $\frac{1}{x}$ approach?

121. Infinity. — Since the series of numbers 1, 2, 3, 4, 5, ⋯, may be continued indefinitely, if a variable x takes in succession all of these values, there is no final value that x may reach. It *increases without limit.* It may thus become greater than any fixed or constant number, however great.

If a variable becomes and remains greater than any constant number, however great, it is said to become **infinite**. A variable that has become infinite is called **infinity**. It is represented by the symbol ∞.

Thus, time is infinite. It extends without limit into the past, and without limit into the future. Similarly space is infinite. It reaches without limit in every direction.

Care has to be used in dealing with infinity in mathematical work. The symbol ∞ may not be operated upon as ordinary numbers may be, because, since it denotes a quantity greater than any fixed number, however great, it may represent widely different values.

For example, the sum of the series of whole numbers, 1, 2, 3, 4, 5, 6, 7, $\cdots$, is infinite, since it is greater than any fixed number that may be assigned. But the sum of the odd numbers, 1, 3, 5, 7, $\cdots$, alone is infinite also. Hence each sum is represented by ∞. Yet one sum must be greater than the other, because the whole of a thing is greater than a part of it.

Again, if we add a number to infinity we get infinity. Thus, $\infty + 6 = \infty$.

122. Values of $\frac{x}{n}$ when $x = \infty$, and when $x \doteq 0$. — If x in the fraction $\frac{1}{x}$ takes in succession the values 1, 2, 3, 4, $\cdots$, without limit there is no final value that $\frac{1}{x}$ may reach, but it becomes smaller and smaller as x takes larger and larger values. As x becomes infinite, $\frac{1}{x}$ approaches zero as a limit. Similarly, in general,

If n in the fraction $\frac{n}{x}$ is any constant not zero, and x a variable which increases without limit, then $\frac{n}{x}$ approaches zero as a limit.

That is, if $x = \infty$, then $\frac{n}{x} \doteq 0$.

Again, if x in the fraction $\frac{1}{x}$ takes in succession the values $\frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \cdots$, without limit, $\frac{1}{x}$ takes in succession larger and larger values and becomes infinite. And in general,

If n in the fraction $\frac{n}{x}$ is any constant not zero, and x a variable which approaches zero as a limit, then $\frac{n}{x}$ increases without limit.

That is, if $x \doteq 0$, then $\frac{n}{x} = \infty$.

123. Meanings of $\frac{n}{0}$ and $\frac{0}{0}$.—Zero may be written as the difference between two equal numbers. Thus $0 = k - k$.

Then if a is any constant,

$$\begin{aligned} a \times 0 &= a(k-k) \\ &= ak - ak \\ &= 0. \end{aligned}$$

Hence the product of any constant multiplied by zero equals zero.

Now the fraction $\frac{n}{0}$, where n is any constant number not zero, represents a number which multiplied by 0 gives n. But, by the above principle, no constant multiplied by 0 gives n. Hence,

The symbol $\frac{n}{0}$ has no meaning.

NOTE.—One sometimes sees the expression $\frac{n}{0} = \infty$. But in this expression the symbol 0 is not to be regarded as absolute zero. The expression $\frac{n}{0} = \infty$ is a loose way sometimes used of saying that $\frac{n}{x} = \infty$ when $x \doteq 0$.

The symbol $\frac{0}{0}$ represents a number which multiplied by 0 gives 0. But, by the above principle, any constant a multiplied by 0 gives 0. Hence,

The symbol $\frac{0}{0}$ represents any constant number whatever.

124. Division by Zero. Since $\frac{0}{0}$ may equal any constant whatever, and $\frac{n}{0}$ has no meaning, evidently,

It is never allowable to divide by zero.

If it were allowable to divide by zero, one could prove most any absurd relation between numbers, as the following example illustrates. Point out the fallacy in the reasoning.

Let $a = b.$

Multiplying by a, $a^2 = ab$

Subtracting b^2, $a^2 - b^2 = ab - b^2.$

Factoring, $(a + b)(a - b) = b(a - b).$

Dividing by $a - b$, $a + b = b.$

Substituting a for b, $2a = a.$

Dividing by a, $2 = 1.$

125. Indeterminate Fractions. — For certain values of the literal number involved, a fraction will sometimes take the **indeterminate form** $\frac{0}{0}$.

For example, when $a = 2$, $\frac{a^2 - 4}{a - 2} = \frac{0}{0}$. This fraction, however, is equal to $\frac{(a + 2)(a - 2)}{a - 2}$. Now, as long as a is not equal to 2, we may divide both terms by $a - 2$. Hence, when a is not equal to 2, $\frac{(a + 2)(a - 2)}{a - 2} = a + 2$. This is true, however little a may differ from 2. Hence, if without giving a the value 2, we make it approach 2 as a limit, we may divide both terms of the fraction by $a - 2$. Now, as a approaches 2, evidently $a + 2$ approaches 4. Hence, we say $\frac{a^2 - 4}{a - 2} \doteq 4$ as $a \doteq 2$. Hence, 4 is called the *value* of $\frac{a^2 - 4}{a - 2}$ when $a = 2$.

The *value* of such a fraction, for any particular value of the literal number involved, is defined as the *limit* which the fraction approaches when the literal number approaches this particular value as its limit.

EXAMPLE. — Find the value of $\frac{x^3 - 27}{x - 3}$ when $x = 3$.

When $x = 3$, $\quad \dfrac{x^3 - 27}{x - 3} = \dfrac{0}{0}$, the indeterminate form.

For any value of x except 3, $\quad \dfrac{x^3 - 27}{x - 3} = x^2 + 3x + 9$.

Now, as $x \doteq 3$, $\quad x^2 + 3x + 9 \doteq 27$.

Hence, $\quad \dfrac{x^3 - 27}{x - 3} = 27$ when $x = 3$.

EXERCISES.

Find the value of:

1. $\dfrac{1 - t}{1 - t^2}$ when $t = 1$.

2. $\dfrac{n - 2}{n^3 - 8}$ when $n = 2$.

3. $\dfrac{x^2 - 4}{x^2 - 5x + 6}$ when $x = 2$.

4. $\dfrac{a^2 + 5a}{a^2 + 3a - 10}$ when $a = -5$

Solve by determinants and interpret solutions:*

5. $\begin{cases} x - y = 2, \\ 3x = 3y + 16. \end{cases}$

6. $\begin{cases} 2m + y = 8, \\ 2m = 1 - y. \end{cases}$

7. $\begin{cases} 9A - 2B = 2, \\ 3A = 5 + \frac{2}{3}B. \end{cases}$

8. $\begin{cases} m + n = 6, \\ 2m = 12 - 2n. \end{cases}$

9. $\begin{cases} 5v - t = 3, \\ 2t = 10v - 6. \end{cases}$

10. $\begin{cases} W + 2V = 20. \\ 3W + 6V = 60 \end{cases}$

SUPPLEMENTARY EXERCISES

1. If $\dfrac{a}{b} = \dfrac{c}{d}$, show that $\dfrac{5a - 2b}{5c - 2d} = \dfrac{\sqrt[3]{a^3 - 4b^3}}{\sqrt[3]{c^3 - 4d^3}}$.

2. If $\dfrac{a}{b} = \dfrac{c}{d}$, show that $\dfrac{a^2 - ab + b^2}{a^2 + ab + b^2} = \dfrac{c^2 - cd + d^2}{c^2 + cd + d^2}$.

3. The surface of a sphere is $4\pi R^2$, where R is the radius. If S denotes the surface of a sphere and D its diameter, show that

$$\frac{S_1}{S_2} = \frac{R_1^2}{R_2^2} = \frac{D_1^2}{D_2^2}.$$

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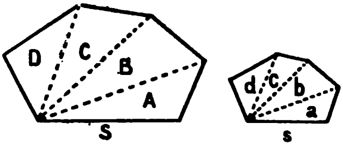
* See § 59. If § 59 was omitted, omit these exercises.



4. Two similar polygons whose corresponding sides are S and s, are divided into similar triangles whose areas are A and a, B and b, C and c, D and d, respectively. It is known that

$$\frac{A}{a}=\frac{S^2}{s^2},\ \frac{B}{b}=\frac{S^2}{s^2},\ \frac{C}{c}=\frac{S^2}{s^2} \text{ and } \frac{D}{d}=\frac{S^2}{s^2}.$$

Show that
$$\frac{A+B+C+D}{a+b+c+d}=\frac{S^2}{s^2}.$$

5. Find the mean proportional between $(m^3-n^3)(m^2+mn+n^2)$ and $m-n$.

6. Find the fourth proportional to 1, $\sqrt{2}$, and $\sqrt{6}$.

Representing the constants involved by k, write the equations which express the following physical laws:

7. The force with which a moving body having a given velocity strikes a stationary body varies directly as the mass or weight of the moving body.

8. The force of attraction between two bodies, such as the earth and the sun, varies directly as the product of their masses, and inversely as the square of the distance between them.

9. If an object is whirled in a circle at the end of a string, the tension or stretching force exerted upon the string by the object varies directly as the product of the weight of the object and the square of its velocity, and inversely as the length of the string.

10. The amount of heat received by a body by radiation from a stove varies inversely as the square of the distance of the body from the stove.

11. The resistance offered by a wire to a current of electricity varies directly as the length of the wire and inversely as the area of its cross section.

Find the limit of each of the following when $n = \infty$:

12. $\frac{1}{n}$. 13. $\frac{5}{n}$. 14. $\frac{1}{n+2}$. 15. $\frac{n}{100}$.

Find the limit of each of the following when $n \doteq 0$:

16. $\frac{1}{n}$. 17. $\frac{n^2}{n^2}$. 18. $\frac{n}{n+1}$. 19. $\frac{n}{n^2+4n}$.

NOTE. — It is easily seen that the symbol $\frac{\infty}{\infty}$ is indeterminate, since the symbol ∞ does not denote a definite value. A fraction which takes this form for some special value of the literal number involved may sometimes be changed to an equivalent fraction which is not of the indeterminate form, and then evaluated, as shown in the following.

Find the limit of each of the following when $n = \infty$:

20. $\frac{n+1}{n+8}$.

SOLUTION. — Dividing both terms by n, $\frac{n+1}{n+8} = \frac{1+\frac{1}{n}}{1+\frac{8}{n}}$.

But when $n = \infty$, $\frac{1}{n} \doteq 0$ and $\frac{8}{n} \doteq 0$.

Hence, when $n = \infty$, $\frac{1+\frac{1}{n}}{1+\frac{8}{n}} \doteq \frac{1}{1}$ or 1.

21. $\frac{n}{n+1}$.

22. $\frac{2n}{n-2}$.

23. $\frac{n+1}{2n}$.

24. $\frac{n+2}{3n-2}$.

25. $\frac{n^2+4}{n^2+n}$.

26. $\frac{2n^2+5n}{4n^2-1}$.

27. $\frac{n(n+1)(n+2)}{n^3}$.

28. $\frac{6n^4-n^2}{(n^2-1)(n^2+3)}$.

CHAPTER XII

PROGRESSIONS

126. Series. — Often, in mathematical work, groups of three or more numbers are encountered that are arranged in a definite order or succession, and that are such that each number of the group may be obtained from one or more of the preceding numbers by some fixed law. Such a group or succession of numbers is called a **series.** The numbers are called the **terms** of the series, and are named from the left to the right, first term, second term, etc.

For example, the odd numbers, 1, 3, 5, 7, ⋯, form a series in which each term after the first is obtained by adding 2 to the preceding term.

Also, 2, 5, 11, 23, ⋯, form a series in which each term after the first is obtained by multiplying the preceding term by 2 and then adding 1 to the product.

And $1, -x, x^2, -x^3, \cdots$, form a series in which each term after the first is obtained by multiplying the preceding term by $-x$.

The law by which the terms of a series are obtained usually may be determined easily by examining the first few terms.

EXERCISES

Give the next 4 terms of each of the following:

1. 1, 2, 3, 4, ⋯.
2. 2, 4, 6, 8, ⋯.
3. 20, 17, 14, ⋯.
4. 1, 2, 4, 8, ⋯.
5. $1, \frac{1}{3}, \frac{1}{9}, \frac{1}{27}, \cdots$.
6. 1, 4, 9, 16, ⋯
7. 3, − 2, − 7, ⋯.
8. $\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \cdots$.
9. 1, 2, 5, 14, ⋯.
10. $1, 2n, 4n^2, 8n^3, \cdots$
11. $\sqrt{2}, \sqrt{5}, \sqrt{8}, \sqrt{11}, \cdots$.
12. $256t^{16}, -128at^{14}, 64a^2t^{12}, \cdots$.

127. Progressions. — Of the many kinds of series, we shall discuss in this chapter only two of the simplest. They are called **progressions.**

128. Arithmetical Progressions. — An **arithmetical progression** is a series in which the difference between any term and the preceding term is the same for all terms.

This difference is called the **common difference.**

The name arithmetical progression is abbreviated usually to A. P.

Thus, the series 1, 4, 7, 10, ⋯, is an A. P. in which the common difference is 3.

The series 10, 5, 0, − 5, − 10, ⋯, is an A. P. in which the common difference is − 5.

129. The *n*th Term of an Arithmetical Progression. — If the first term of any arithmetical progression is denoted by a, and the common difference by d, the terms of the progression will be represented by

$$a,\ a+d,\ a+2\,d,\ a+3\,d,\ a+4\,d,\ a+5\,d, \text{ etc.}$$

It is evident from these expressions that the coefficient of d in any term is 1 less than the number of the term. Thus, the twelfth term is $a+11\,d$. Hence, the general or nth term is $a+(n-1)d$. Therefore, if l represents the value of the nth term in an arithmetical progression whose first term is a and common difference d,

$$\boldsymbol{l=a+(n-1)d}. \qquad (A)$$

This formula will enable one to find the value of any one of the four numbers, l, a, n, d, if the values of the other three are known.

EXAMPLE 1. — Find the twentieth term of the series 4, 7, 10, 13, ⋯.

Here $a=4$, $d=3$, and $n=20$.

Substituting in formula A,

$$l=4+(20-1)3=61.$$

Example 2. — In the series 5, 9, 13, 17, ⋯, what term of the series is 65?

Here, $a = 5$, $d = 4$, $l = 65$, n is unknown.

Substituting in formula A,

$$65 = 5 + (n - 1)4.$$

Solving,
$$n = 16.$$

Hence, 65 is the sixteenth term of the series.

Example 3. — The fourth and fifteenth terms of an A. P. are 9 and 31, respectively. Write the first six terms.

The first term and common difference must first be found. Since the fourth term is $a + 3d$, and the fifteenth term is $a + 14d$, we get the system

$$\begin{cases} a + 3d = 9, \\ a + 14d = 31. \end{cases}$$

Solving this system, $a = 3$, and $d = 2$.

Hence, the series is 3, 5, 7, 9, 11, 13, ⋯.

EXERCISES

1. Find the thirtieth term of the series 1, 6, 11, 16, ⋯.

2. Find the sixteenth term of the series 1, 5, 9, 13, ⋯.

3. Find the twenty-fourth term of the series -8, -5, -2, 1, ⋯.

4. Find the twentieth term of the series 12, 6, 0, -6, ⋯.

5. Find the eighteenth term of the series $-\frac{3}{2}$, $-\frac{1}{2}$, $\frac{1}{2}$, $\frac{3}{2}$, ⋯.

6. Find the forty-second term of the series 11, 17, 23, ⋯.

7. Which term of the series 1, 4, 7, 10, ⋯, is 46?

8. Which term of the series 2, 7, 12, 17, ⋯, is 152?

9. Which term of the series 12, 8, 4, 0, ⋯, is -132?

10. In an A. P. whose common difference is 3, the sixteenth term is 48. Find the first term.

11. In an A. P. whose common difference is -2, the twenty-eighth term is -49. Find the first term.

12. In an A. P. whose first term is 7, the twenty-second term is 175. Find the common difference.

13. In an A.P. whose first term is 12, the thirtieth term is 99. Find the sixteenth term.

14. The sixth term of an A. P. is 17, and the fifteenth term is 44. Find the first term and the common difference.

15. The third term of an A. P. is 0, and the ninth term is 22. Find the first term and the common difference.

16 The fifth term of an A.P. is 21, and the eighth term is 33. Find the twelfth term. The twentieth term.

17. The second term of an A. P. is 7, and the eleventh term is $20\frac{1}{2}$. Find the seventh term. The fifteenth term.

18. Beginning with 2, find the 100th even number.

19. Beginning with 1, find the 100th odd number.

20. Beginning with 5×4, or 20, find the 120th multiple of 4.

21. Find the twenty-fifth term of the series $n, 4n, 7n, \cdots$.

22. Find the sixteenth term of the series $1, 1 + 3x, 1 + 6x, \cdots$.

23. Find the $(n - 1)$st term of the series $a, a + d, a + 2d, \cdots$.

24. Find the $(n - 3)$d term of the series $a, a + d, a + 2d, \cdots$.

25. A body falls 16 feet the first second, 48 feet the second, 80 feet the third, etc. How far does it fall during the 20th second? During the 60th second?

26. A body falls $\frac{1}{2}a$ feet the first second, $\frac{3}{2}a$ feet the next second, $\frac{5}{2}a$ feet the next second, etc. How far does it fall during the tth second?

27. A projectile is thrown vertically upward with an initial velocity of 448 feet per second. At the end of 1 second its velocity is 416 feet per second, at the end of 2 seconds its velocity is 384 feet per second, etc. In how many seconds will it stop rising and begin to descend?

130. Arithmetical Means. — In an arithmetical progression all of the terms between any two terms are called the **arithmetical means** of those two terms.

For example, since 2, 4, 6, 8, and 10 form an A. P., 4, 6, and 8 are arithmetical means of 2 and 10.

Any number of arithmetical means may be inserted between two given numbers.

Example. — Insert seven arithmetical means, between 10 and 30.

Since there are to be seven arithmetical means, 30 must be the ninth term of an A. P. of which 10 is the first term.

Hence, we have $a = 10$, $l = 30$, $n = 9$.

Substituting in formula A,

$$30 = 10 + 8d.$$

Solving, $d = 2\frac{1}{2}$.

Hence, the means are $12\frac{1}{2}$, 15, $17\frac{1}{2}$, 20, $22\frac{1}{2}$, 25, $27\frac{1}{2}$.

A very important special case is that of one **arithmetical mean** between two numbers. It is called also the **average** of the two numbers.

If A is the arithmetical mean between a and b, then by definition of an A. P.,

$$A - a = b - A.$$

Solving for A,

$$A = \frac{a + b}{2}.$$

Hence the arithmetical mean between two numbers equals one half of their sum.

EXERCISES

1. Insert four arithmetical means between 5 and 25.
2. Insert three arithmetical means between 12 and -2.
3. Insert five arithmetical means between 3 and 27.
4. Insert six arithmetical means between -6 and 29.
5. Insert six arithmetical means between $\frac{1}{2}$ and -10.
6. Insert twelve arithmetical means between 1 and 5.
7. Insert twenty arithmetical means between 3 and 48.
8. Insert two arithmetical means between $\frac{1}{2}$ and $\frac{1}{3}$.
9. Insert five arithmetical means between n and $\frac{1}{6}n$

10. Find the arithmetical mean of 5 and 19. Of 8 and 72. Of 16 and 34. Of $\frac{1}{2}$ and $3\frac{1}{2}$. Of -6 and 2. Of $\frac{1}{8}$ and $1\frac{1}{2}$. Of $-\frac{2}{3}$ and $\frac{5}{6}$. Of $2\sqrt{3}$ and $\sqrt{3}$. Of 1 and $2x-1$.

11. Find the arithmetical mean of a^2+ab+b^2 and a^2-ab+b^2.

12 The velocity of a falling body increases uniformly throughout its descent. Its velocity at the beginning of the second second is 32 feet per second, and at the end of the second second 64 feet per second. Find the average or mean velocity during the second second.

13. The velocities of a falling body at the beginning and end of the fifth second are 160 feet per second and 192 feet per second, respectively. Find the average velocity during the fifth second.

14. In going a distance of 20 miles an aviator increased his speed uniformly from 24 miles per hour to 40 miles per hour. What was his average speed for the trip? How long did it take to make the trip?

15. A certain distance is divided into 6 parts, which are in arithmetical progression. The shortest part is 2 inches and the longest 56 inches. Find the lengths of the other parts.

16. A set of 8 weights are in arithmetical progression. The smallest is 1 ounce and the largest 15 ounces. Find the other weights.

17. If the temperature at 6 A.M. is $-4°$, and rises uniformly 2° per hour until 2 P.M., what is the average temperature for the entire period?

131. Sum of n Terms of an Arithmetical Progression. — The sum of n terms of an arithmetical progression may be obtained as follows:

If the sum be represented by S, then

$$S=a+(a+d)+(a+2d)+(a+3d)+\cdots+(l-d)+l$$

If the terms are written in the reverse order,

$$S = l + (l - d) + (l - 2d) + (l - 3d) + \cdots + (a + d) + a.$$

Adding the corresponding terms of these two equations, the d's are eliminated, and we get

$$2S = (a + l) + (a + l) + (a + l) + \cdots \text{to } n \text{ terms}$$
$$= n(a + l).$$

Hence,

$$S = \tfrac{1}{2}n(a + l). \qquad (B)$$

Substituting in this formula the value of l given in formula A § 129, we get another formula:

$$S = \tfrac{1}{2}n\{2a + (n - 1)d\}. \qquad (C)$$

If the value of d is not known and the values of n, a, and l are known, S may be found by formula B. If the value of l is not known, and the values of n, a, and d are known, S may be found by formula C.

If any three of the five numbers a, d, n, l, S, of an A. P. are known, the values of the other two may be found by use of formulas A, B, and C.

Example 1. — Find the sum of 25 terms of the series -9, -5, -1, 3, 7, ….

Here $a = -9$, $d = 4$, $n = 25$. Hence, substituting in formula C,

$$S = \tfrac{1}{2} \times 25\{2(-9) + (25 - 1)4\} = 975.$$

Example 2. — The first term of an A. P. is 2, and the sum of 20 terms is 135. Find the common difference.

Here $a = 2$, $n = 20$, $S = 135$. Hence, from formula C,

$$135 = 10(4 + 19d).$$

Solving,

$$d = \tfrac{1}{2}.$$

Example 3. — How many terms of the series 3, 1, -1, -3, -5, …, must be taken to make -140?

Here $a = 3$, $d = -2$, $S = -140$. Hence, from formula C,

$$-140 = \tfrac{1}{2}n\{6 + (n - 1)(-2)\}.$$

Simplifying, $n^2 - 4n - 140 = 0.$

Solving, $n = 14 \text{ or } -10.$

The root -10 has no meaning, and hence must be discarded. Therefore 14 terms must be taken.

EXAMPLE 4. — The common difference in an A. P. is $1\frac{1}{2}$, and the twelfth term is $12\frac{1}{2}$. Find the sum of the first 7 terms.

From formula *A*, $a = -4.$

Hence, from formula *C*, $S = 3\frac{1}{2}.$

EXERCISES

1. Find the sum of 25 terms of the series 3, 8, 13, ⋯.

2. Find the sum of 16 terms of the series 8, 12, 16, ⋯.

3. Find the sum of 22 terms of the series 42, 39, 36, ⋯.

4. Find the sum of 30 terms of the series 12, 9, 6, ⋯.

5. Find the sum of 40 terms of the series $\frac{1}{2}$, $\frac{5}{8}$, $\frac{3}{4}$, ⋯.

6. Find the sum of 36 terms of the series $12\frac{1}{2}$, 8, $3\frac{1}{2}$,

7. Find the sum of the first 100 whole numbers.

8. Find the sum of the first 100 even numbers.

9. Find the sum of the first 100 odd numbers.

10. Find the sum of all odd numbers of two digits.

11. Find the sum of all even numbers between 100 and 200.

12. Find the sum of all numbers between 50 and 250 that are divisible by 4.

13. Show that the sum of the first n odd numbers is n^2.

14. Find the sum of the first n even numbers.

15. How many of the series of whole numbers, beginning with 1, must be added to give 3240?

16. How many terms of the series 15, 12, 9, ⋯, must be added to give 45?

17. How many terms of the series $6\frac{1}{2}$, $4\frac{1}{2}$, $2\frac{1}{2}$, ⋯, must be added to give -396?

18. How many terms of the series a, $3a$, $5a$, $\cdots$, must be added to give $400a$?

19. If $a = 4$, $d = 3$, $l = 40$, find n and S.

20. If $a = 2$, $d = 5$, $S = 1428$, find n and l.

21. If $d = -4$, $n = 18$, $S = 144$, find a and l.

22. The first term of an A.P. is -8, and the sum of 10 terms is 100. Write the first 5 terms of the series.

23. If a common clock strikes only the hours, how many strokes does it make in a day?

24. If the hours of the day are numbered from 1 to 24, as in Italy, how many strokes would a clock that strikes only the hours make in a day?

25. In a potato race 50 potatoes are placed in a straight line, 1 yard apart, the first one being 1 yard from the basket. How far must a contestant run to bring them to the basket one at a time?

26. If an elastic ball falls from a height of 30 inches, and at each rebound rises to a point 6 inches below the point to which it rose the time before, find the entire distance that it travels before coming to rest.

27. A car starting down an incline moves 1 foot the first second, 4 feet the second second, 7 feet the third second, etc. How far will it move in 20 seconds? How long will it take the car to run 1335 feet, the entire length of the incline?

28. An object falling freely through space falls approximately 16 feet the first second, 48 feet the next, 80 feet the next, etc. How far will it fall in 6 seconds? In 10 seconds? In 60 seconds?

29. A shot is fired vertically upward, going 1024 feet the first second, and lessening its speed 32 feet per second thereafter. How high does the shot go?

30. If an object falls 16 feet the first second, 48 feet the next, 80 feet the next, etc., and an aviator in time of war drops a bomb from a height of 1600 feet, how long does it take it to reach the ground?

31. A man invests his savings in the shares of a building and loan association, depositing with the association $1000 the first year. At the beginning of the second year he is credited with $60 interest on the amount deposited the first year, and pays in only $940, making his total credit $2000. At the beginning of the third year he is credited with $120 interest, and pays in only $880 cash, etc., What is his credit at the end of 10 years, and how much cash has he paid in then?

32. If a man invests $1000 of his savings at the end of each year for 10 years at 5% simple interest, how much will he have at the end of 10 years?

33. Cannon balls are piled up into a pyramid with a square base having 12 balls on each side. How many balls are there in the bottom layer? In the second layer? In the third layer? In the whole pyramid?

34. A roll of paper is wrapped about a core whose diameter is 1 inch. The thickness of the paper is $\frac{1}{64}$ inch. The outer diameter of the roll is 10 inches. How long is the strip of paper in the roll?

35. Find three numbers in A. P., such that their sum is 18 and the product of the first and third 14 greater than the second.

36. The sum of three numbers in A. P. is 15, and the sum of their squares is 107. Find the numbers.

132. Geometrical Progressions. — A *geometrical progression* is a series in which the ratio of any term to the preceding term is the same for all terms of the series.

This ratio is called the **common ratio.**

The name geometrical progression is abbreviated usually to G. P.

Thus, the series 1, 2, 4, 8, $\cdots$, is a G. P., in which the common ratio is 2.

The series $2, -\frac{1}{2}n, \frac{1}{8}n^2, -\frac{1}{32}n^3, \cdots$, is a G. P. in which the common ratio is $-\frac{1}{4}n$.

In either series, if any term is multiplied by the common ratio, the product will be the next term.

133. The nth term of a Geometrical Progression. — If the first term of any geometrical progression is denoted by a, and the common ratio by r, the terms of the progression will be represented by

$$a,\ ar,\ ar^2,\ ar^3,\ ar^4,\ ar^5,\ \text{etc.}$$

It is seen from these terms that the power of r in any term is 1 less than the number of the term. Thus, the tenth term is ar^9. Hence, the general or nth term is ar^{n-1}. Therefore, if l denotes the value of the nth term in the geometrical progression whose first term is a and common ratio r,

$$\mathbf{l = ar^{n-1}}. \qquad (A)$$

This formula will enable one to find the value of any one of the four numbers l, a, r, n, if the values of the other three are known.

EXAMPLE 1. — Find the ninth term of the series 2, 6, 18, $\cdots$.

Here $a = 2,\ r = 3,\ n = 9$.

Substituting in formula A,

$$l = 2 \times 3^8 = 13{,}122.$$

EXAMPLE 2. — The tenth term of a G. P. is $\frac{1}{512}$, and the first term is 1. Find the common ratio.

Here $a = 1,\ n = 10,\ l = \frac{1}{512}$.

Substituting in formula A,

$$\tfrac{1}{512} = 1 \times r^9.$$

Hence, $r = \frac{1}{2}$.

EXAMPLE 3. — The fifth and eighth terms of a G.P. are $\frac{16}{27}$ and $-\frac{128}{729}$, respectively. Write the first 8 terms.

Since the fifth term is ar^4, and the eighth term is ar^7,

$$\begin{cases} ar^4 = \frac{16}{27}, & (1) \\ ar^7 = -\frac{128}{729}. & (2) \end{cases}$$

Dividing (2) by (1), $r^3 = -\frac{8}{27}$.

Hence, $r = -\frac{2}{3}$.

Substituting $-\frac{2}{3}$ for r in (1), $\frac{16}{81}a = \frac{16}{27}$.

Hence, $a = 3$.

Therefore the series is $3,\ -2,\ \frac{4}{3},\ -\frac{8}{9},\ \frac{16}{27},\ -\frac{32}{81},\ \frac{64}{243},\ -\frac{128}{729},\ \cdots$.

EXERCISES

1. Find the eighth term of the series $1, 2, 4, \cdots$.

2. Find the twelfth term of the series $\frac{1}{1000}, \frac{1}{100}, \frac{1}{10}, \cdots$

3. Find the tenth term of the series $2430, 810, 270, \cdots$

4. Find the eighth term of the series $\frac{2}{3}, 2, 6, \cdots$.

5. Find the ninth term of the series $4, -8, 16, \cdots$.

6. Find the sixth term of the series $54, -18, 6, \cdots$.

7. Find the tenth term of the series $\frac{1}{16}, \frac{1}{4}, 1, \cdots$.

8. The first term of a G. P. is 4, and the sixth term is 128. Find the common ratio.

9. The common ratio of a G. P. is 3, and the fifth term is 486. Find the first term.

10. The second term of a G. P. is $\frac{1}{4}$, and the eighth term is $\frac{1}{256}$. Find the tenth term.

11. The first term of a G. P. is 3, and the third term is 6. Find the sixth term.

12. The second term of a G. P. is 3, and the fifth term is $\frac{81}{8}$. Find the fourth term.

13. The eighth term of a G. P. is 1152, and the tenth term is 4608. Find the first five terms.

14. The third term of a G. P. is $\frac{1}{8}$, and the eighth term is 128. Find the first six terms.

15. Find the sixth term of the series $2\sqrt{2}, 4, 4\sqrt{2}, \cdots$.

16. Find the fifth term of the series $\dfrac{2}{3\sqrt{2}}, 1, \dfrac{3\sqrt{2}}{2}, \cdots$.

17. The first and second terms of a G. P. are m and n. Find the next two terms.

18. In making an inventory at the close of each year, a manufacturer deducted 10% from the value of his machinery at the previous inventory, because of deterioration. The machinery cost $20,000. What was the value at the end of the fifth year?

SUGGESTION. — The value at the end of the first year was $.90 \times \$20,000$; at the end of the second year, $.90 \times .90 \times \$20,000$, or $.90^2 \times \$20,000$; etc. What was the common ratio?

19. A boy puts $100 in a savings bank, which pays 3 % compound interest, compounded annually. What does it amount to at the end of six years?

SUGGESTION. — The value at the end of the first year is $1.03 \times \$100$; at the end of the second year, $1.03^2 \times \$100$; etc.

20. To how much will $500 amount at the end of five years, if put in a savings bank which pays 3 % compound interest, compounded annually?

21. The population of a city is 100,000, and it increases 10 % every year. What will the population be in 10 years?

22. If a rubber ball rebounds $\frac{1}{4}$ of the distance it falls each time, and falls the first time from a height of 60 feet, how high will it rise after the eighth fall?

134. Geometrical Means. — In a geometrical progression all of the terms between any two terms are called the **geometrical means** of those two terms.

For example, since 5, 10, 20, 40, 80, and 160 form a G. P., 10, 20, 40, and 80 are geometrical means of 5 and 160.

Any number of geometrical means may be inserted between two given numbers.

EXAMPLE. — Insert two geometrical means between 8 and 27.

Since there are to be two geometrical means, 27 must be the fourth term of a G. P. of which 8 is the first term.

Hence, we have $a = 8$, $l = 27$, and $n = 4$.

Substituting in formula A,

$$27 = 8r^3.$$

Solving,

$$r = 1\tfrac{1}{2}.$$

Hence, the means are 12 and 18.

An important special case is that of one **geometrical mean** between two numbers. It is called also the **mean proportional** between the two numbers.

If G is the geometrical mean between a and b, then by definition of a G.P.,

$$\frac{b}{G}=\frac{G}{a}.$$

Solving for G, $G=\sqrt{ab}$.

Hence, the geometrical mean between two numbers equals the square root of their product.

EXERCISES

1. Insert two geometrical means between 7 and 56.

2. Insert two geometrical means between 2 and 54.

3. Insert three geometrical means between 16 and 54.

4. Insert four geometrical means between 1 and 1024.

5. Insert three geometrical means between 80 and 5.

6. Insert four geometrical means between 24 and $3\frac{13}{81}$.

7. Insert three geometrical means between -9 and -144.

8. Insert two geometrical means between 1 and 2.

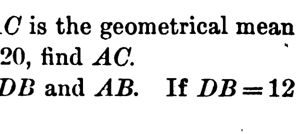
9. Insert two geometrical means between m and n.

10. Find the gometrical mean of 1 and 4. Of 9 and 16. Of 5 and 20. Of 3 and 4. Of $\sqrt{2}$ and $\sqrt{8}$.

11. Find the geometrical mean of $a-b$ and a^3-a^2b.

12. Triangle ABC is a right triangle, with CD perpendicular to the hypotenuse AB. It is shown in geometry that CD is the geometrical mean of AD and DB.

If $AD=9$ and $DB=25$, find CD.
If $AD=4$ and $DB=12$, find CD.

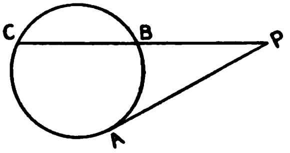


13. In the figure of Ex. 12, if $CD=8$ and $AB=20$, find AD and DB.

14. In the right triangle in Ex. 12, AC is the geometrical mean of AD and AB. If $AD=5$ and $AB=20$, find AC.

Also CB is the geometrical mean of DB and AB. If $DB=12$ and $AB=27$, find CB.

15. If PA touches a circle at A, and PB cuts the circle at B and C, PA is a geometrical mean of PB and PC.



If $PB = 16$ and $PC = 49$, find PA.

If $BC = 12$ and $PA = 16$, find PB and PC.

135. Sum of n Terms of a Geometrical Progression. — The sum of n terms of a geometrical progression is found as follows:

If S represents the sum of n terms of a geometrical progression whose first term is a and common ratio r, then

$$S = a + ar + ar^2 + ar^3 + \cdots + ar^{n-2} + ar^{n-1}. \qquad (1)$$

Hence, $$rS = ar + ar^2 + ar^3 + ar^4 + \cdots + ar^{n-1} + ar^n. \qquad (2)$$

Now, if (2) be subtracted from (1), all terms cancel except a of (1) and ar^n of (2). Hence, subtracting (2) from (1),

$$S - rS = a - ar^n.$$

Factoring, $$S(1 - r) = a(1 - r^n).$$

Hence, dividing by $1 - r$, we get

$$\mathbf{S = \frac{a(1 - r^n)}{1 - r}}. \qquad (B)$$

Since $l = ar^{n-1}$, by substituting in (B) we get

$$\mathbf{S = \frac{a - rl}{1 - r}}. \qquad (C)$$

If any three of the numbers a, r, n, l, S, of a G. P. are known, the other two may be found by use of formulas A, B, and C.

Example 1. — Find the sum of the first 6 terms of the series 4, 6, 9, ⋯.

Here $$a = 4,\ r = \tfrac{3}{2},\ n = 6.$$

Hence, $$S = \frac{4\{1 - (\frac{3}{2})^6\}}{1 - \frac{3}{2}} = 83\tfrac{1}{8}.$$

Example 2. — The first term of a G. P. is 3, the sixth term is 9375, and the sum of the first 6 terms is 11,718. Find the common ratio.

By formula C, $$11{,}718 = \frac{3 - 9375\,r}{1 - r}.$$

Solving, $$r = 5.$$

EXERCISES

1. Find the sum of 10 terms of 6, 12, 24, ⋯.

2. Find the sum of 6 terms of 1, 3, 9, ⋯.

3. Find the sum of 7 terms of 2, -4, 8, ⋯.

4. Find the sum of 8 terms of $\frac{1}{16}$, $\frac{1}{8}$, $\frac{1}{4}$, ⋯.

5. Find the sum of 6 terms of 48, 12, 3, ⋯.

6. Find the sum of 5 terms of 5, -30, 180, ⋯.

7. Find the sum of 10 terms of 1, x, x^2, ⋯.

8. Find the sum of 9 terms of $\frac{2}{3}$, 1, $\frac{3}{2}$, ⋯.

9. Find the sum of 6 terms of $6n$, $-12n^2$, $24n^3$, ⋯.

10. Find the sum of 5 terms of $2\sqrt{3}$, 6, $6\sqrt{3}$, ⋯.

11. Find the sum of 10 terms of $\frac{4}{3}$, $\frac{2}{3}$, $\frac{1}{3}$, ⋯.

12. In a G. P. whose first term is 7, the sum of 6 terms is 2548. Find the common ratio.

13. In a G. P. whose first term is $\frac{1}{4}$, the sum of 5 terms is $\frac{211}{324}$. Find the common ratio. Find the fifth term.

14. In the series 1, 2, 4, ⋯, how many terms must be added to make 255?

15. In the series 100, 40, 16, ⋯, how many terms must be added to make $\frac{20748}{125}$?

16. A ball falls from a height of 100 feet, and rebounds after each fall one fifth of the distance it fell. Through what distance will it have traveled at the end of the fifth fall?

17. The bell jar of an air pump holds 432 cubic inches of air. At each stroke of the pump $\frac{1}{12}$ of the air left in the jar is removed. How much air is removed in 5 strokes?

18. If a man saves each year twice as much as the preceding year, and saves $150 the first year, how long will it take for him to save $2250?

19. A vessel containing vinegar is emptied of one-half its contents, and then filled with water. Half of this mixture is then drawn off and the vessel refilled with water. This is repeated five times. What part of the remaining mixture is water?

20. To what sum will \$1 amount in 4 years at 3 % compound interest?

SUGGESTION. — $a = \$1$, $r = \$1.03$, $n = 5$.

21. To what sum will \$1000 amount in 5 years at 3 % compound interest?

22. A man puts \$100 in a savings bank which pays 3 % compound interest, compounded annually. To what sum will it amount in 6 years?

23. The **present worth** of a debt due at some future time is a sum such that, if invested at compound interest, the amount at the end of the time will equal the debt.

Find the present worth of \$5000 due in 5 years, the money being worth 6 % interest compounded annually.

24. A man, in making his will, left \$25,000 to his daughter. It was made payable on her twenty-fifth birthday, but in case she married before that time she was to receive the present worth of the sum at the time of her marriage, money being considered worth 6 % compounded annually. She married on her twentieth birthday. How much money did she receive?

136. Infinite Geometrical Progressions. — In a G. P. such as 1, 2, 4, 8, ⋯, in which the common ratio is greater than 1, and positive, each term is greater than the preceding term. Evidently, if the number of terms is increased without limit, their sum will become infinite.

But in a G. P. such as 1, $\frac{1}{2}$, $\frac{1}{4}$, $\frac{1}{8}$, ⋯, in which the common ratio is less than 1, and positive, each term is less than the preceding term, and as the number of terms becomes very great each term becomes very small. In such a series, as the number of terms taken is increased without limit, the sum of the terms does not increase without limit, but approaches some definite number as a limit.

For example, in the series 1, $\frac{1}{2}$, $\frac{1}{4}$, $\frac{1}{8}$, ⋯, the sum of 4 terms is $1\frac{7}{8}$, the sum of 8 terms is $1\frac{127}{128}$, the sum of 16 terms is $1\frac{32767}{32768}$, etc. Evidently, as the number of terms is gradually increased the sum approaches nearer and nearer to 2, but always remains less than 2. When the number of terms taken is increased without limit, the sum approaches 2 as a limit.

In general, the formula $S=\frac{a(1-r^n)}{1-r}$ may be written in the form $S=\frac{a}{1-r}-\frac{ar^n}{1-r}$. Now, if r is less than 1, r^n is still smaller, and when $n=\infty$, $r^n \doteq 0$. When $r^n \doteq 0$, $\frac{ar^n}{1-r} \doteq 0$. Hence, when n is increased without limit, the sum $\frac{a}{1-r}-\frac{ar^n}{1-r}$ approaches $\frac{a}{1-r}$ as a limit.

Consequently, the sum of an infinite number of terms of a G. P. in which the common ratio is less than 1 *is defined as the limit* $\frac{a}{1-r}$.

That is, when n is infinite and r less than 1,

$$S=\frac{a}{1-r}. \qquad (D)$$

Example. — Find the sum of the infinite series of terms 9, 6, 4, ⋯.

Here $r=\frac{2}{3}$. Hence from formula D,

$$S=\frac{9}{1-\frac{2}{3}}=27.$$

By aid of formula D the common fraction to which any repeating decimal is equal may be found.

Example. — Find the value of 12.636363 ⋯.

The repeating part .636363 ⋯ may be written in the form $\frac{63}{100}+\frac{63}{10000}+\frac{63}{1000000}+\cdots$. This is an infinite G. P. in which $a=\frac{63}{100}$ and $r=\frac{1}{100}$.

Hence, $$.636363\cdots=\frac{\frac{63}{100}}{1-\frac{1}{100}}=\frac{63}{99}=\frac{7}{11}.$$

Hence, $$12.636363\cdots=12\tfrac{7}{11}.$$

EXERCISES

Find the sum of an infinite number of terms of:

1. $8, 4, 2, \cdots$.
2. $12, 4, 1\frac{1}{3}, \cdots$.
3. $15, 5, 1\frac{2}{3}, \cdots$.
4. $3, \frac{1}{4}, \frac{1}{48}, \cdots$.
5. $\frac{3}{2}, 1, \frac{2}{3}, \cdots$.
6. $\frac{1}{2}, \frac{1}{8}, \frac{1}{32}, \cdots$.
7. $1, \frac{1}{5}, \frac{1}{25}, \cdots$.
8. $2, \sqrt{2}, 1, \cdots$.
9. $1, n, n^2, \cdots$, when n is less than 1.
10. $1, \frac{1}{n^2}, \frac{1}{n^4}, \cdots$, when n is greater than 1.

11. In a G. P. the common ratio is $\frac{1}{4}$. What must be the first term in order that the sum of an infinite number of terms shall be 80?

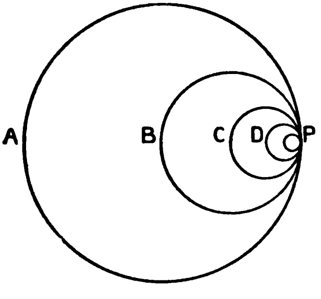
12. In a G. P. the first term is 5. What must be the common ratio in order that the sum of an infinite number of terms shall be $4\frac{6}{11}$?

13. Find the first five terms of a G. P. the sum of an infinite number of terms of which is 2, and whose second term is $\frac{3}{8}$.

14. Find the eighth term of a G. P. the sum of an infinite number of terms of which is 243, and whose second term is 54.

Find the common fractions to which the following are equal:

15. $0.66666 \cdots$.
16. $0.272727 \cdots$.
17. $0.393939 \cdots$.
18. $0.454545 \cdots$.
19. $0.150150 \cdots$.
20. $1.333333 \cdots$.
21. $4.363636 \cdots$.
22. $3.721212 \cdots$.
23. $0.046363 \cdots$.



24. The circles A, B, C, etc., are tangent internally at P. The diameter of B is $\frac{1}{2}$ that of A, the diameter of C is $\frac{1}{2}$ that of B, etc. If the diameter of A is 6 inches, and the number of circles thus constructed is continued without limit, find the sum of the circumferences of all of the circles.

25. In the figure of Ex. 24, the area of circle B is $\frac{1}{4}$ that of circle A, the area of circle C is $\frac{1}{4}$ that of circle B, etc. If the diameter of A is 4 inches, find the sum of the areas of all of the circles.

SUPPLEMENTARY EXERCISES

The numbers a, b, c, d, etc., are said to form a **harmonic progression** when $\frac{1}{a}, \frac{1}{b}, \frac{1}{c}, \frac{1}{d}$, etc., form an arithmetical progression. Problems relating to harmonic progressions may be solved by forming the corresponding arithmetical progressions and using the principles applicable to the latter.

1. Find the twelfth term of the series $1, \frac{1}{3}, \frac{1}{5}, \frac{1}{7}, \cdots$.

SOLUTION. — The corresponding A. P. is $1, 3, 5, 7, \cdots$.
The twelfth term of this is $1 + 11 \times 2$, or 23.
Hence, the twelfth term of the H. P. is $\frac{1}{23}$.

2. Find the seventh term of the series $1, \frac{1}{6}, \frac{1}{11}, \cdots$.

3. Find the twentieth term of the series $-\frac{1}{8}, -\frac{1}{5}, -\frac{1}{2}, \cdots$.

4. Find the fifteenth term of the series $-\frac{2}{3}, -2, 2, \frac{2}{3}, \cdots$.

5. Find the twelfth term of the series $2, \frac{1}{4}, \frac{2}{15}, \cdots$.

6. Find the tenth term of the H. P. in which the third term is $\frac{1}{3}$ and the seventh term is $\frac{1}{83}$.

7. Find the first five terms of the H. P. in which the third term is 2 and the thirty-second term is $\frac{2}{59}$.

8. In any H. P. all of the terms between any two given terms are called the *harmonic means* of those terms.

Find the harmonic mean of $\frac{1}{6}$ and $\frac{1}{12}$. Of $-\frac{3}{2}$ and $\frac{3}{4}$.

9. Insert 4 harmonic means between $-\frac{6}{5}$ and $\frac{6}{5}$.

10. Insert 5 harmonic means between $\frac{1}{4}$ and 3.

11. If H represents the harmonic mean between a and b, show that $H = \frac{2ab}{a+b}$.

12. If A, G, and H represent the arithmetical, geometrical, and harmonic means, respectively, of two numbers, then show that G is a mean proportional between A and H.

Suggestion. — If the two numbers are a and b, then

$$A = \frac{a+b}{2},\ G = \sqrt{ab},\ H = \frac{2ab}{a+b}.$$

Now show that $AH = G^2$, by showing that they equal the same thing.

13. The arithmetical mean of two numbers is 8, and their harmonic mean is 18. Find their geometrical mean.

14. The geometrical mean of two numbers is 4, and their harmonic mean is $\frac{1}{5}$. Find the numbers.

CHAPTER XIII

EXPONENTS

137. Positive Integral Exponents. — A positive integral **exponent**, as defined and used in the FIRST COURSE and in this volume, indicates how many times the base is to be taken as a factor.

The following fundamental laws of positive integral exponents, which have been applied at various points of the work, were proved in Chapter XVI of the FIRST COURSE by use of the above definition. Let the pupil establish each of them in review.

I. LAW FOR MULTIPLICATION :

$$a^m \times a^n = a^{m+n}.$$

That is, *the exponent of the product of two powers of the same number equals the sum of the exponents of the two factors.*

SUGGESTION. — Show by definition of an exponent that $a^m \times a^n$ contains a as a factor $m + n$ times.

This law may be extended to apply to the product of more than two factors. Thus, in general,

$$a^m \times a^n \times a^p \cdots = a^{m+n+p+\cdots}.$$

II. LAW FOR DIVISION :

$$a^m \div a^n = a^{m-n}.$$

That is, *the exponent of the quotient of two powers of the same number equals the exponent of the dividend minus the exponent of the divisor.*

SUGGESTION. — By using the principle that *quotient* × *divisor* = *dividend*, show that this law follows from Law I.

III. LAW FOR A POWER OF A POWER:

$$(a^n)^m = a^{mn}.$$

That is, *the mth power of the nth power of any number equals the mnth power of the number.*

SUGGESTION. — Show that this follows from Law I. Thus,

$$(a^n)^m = a^n \times a^n \times a^n \cdots \text{ to } m \text{ factors, etc.}$$

This law may be extended thus:

$$((a^n)^m)^p = a^{pmn}; \ (((a^n)^m)^p)^q = a^{qpmn}; \text{ etc.}$$

IV. LAW FOR A POWER OF A PRODUCT:

$$(ab)^n = a^n b^n.$$

That is, *the nth power of the product of two numbers equals the product of the nth powers of the numbers.*

SUGGESTION — Show that $(ab)^n$ contains a and b each as a factor n times. Thus,

$$(ab)^n = (ab)(ab)(ab) \cdots \text{ to } n \text{ factors, etc.}$$

This law may be extended to apply to a power of a product of three or more factors. Thus, in general,

$$(abc \cdots)^n = a^n b^n c^n \cdots.$$

V. LAW FOR POWER OF A FRACTION:

$$\left(\frac{a}{b}\right)^n = \frac{a^n}{b^n}.$$

That is, *the nth power of a fraction equals the nth power of the numerator divided by the nth power of the denominator.*

SUGGESTION. — Proceed as in establishing Law IV.

VI. LAW FOR ROOT OF A POWER, OR POWER OF A ROOT:

$$\sqrt[n]{a^m} = a^{\frac{m}{n}}.$$

That is, *the exponent of a root of a power of a number equals the exponent of the power divided by the index of the root.*

SUGGESTION. — $(a^{\frac{m}{n}})^n = a^{\frac{m}{n} \times n} = a^m$, by Law III.

Since by this law $\sqrt[n]{a}=a^{\frac{1}{n}}$, $(\sqrt[n]{a})^m=(a^{\frac{1}{n}})^m$. Hence by Law III,

$$(\sqrt[n]{a})^m=a^{\frac{m}{n}}.$$

That is, *the exponent of a power of a root of a number equals the exponent of the power divided by the index of the root.*

By use of one or more of the above laws of exponents, any power or root of any monomial, or the product or quotient of any monomials, may be obtained.

EXERCISES

By the above laws find the values of:

1. $n^3 \times n^7$.
2. $V^8 \times V^4$.
3. $k \times k^5 \times k^3$.
4. $R^6 \times R \times R^8$.
5. $2^3 \times 2^2 \times 2$.
6. $5 \times 5^2 \times 5^4$.
7. $3^4 \times 3 \times 3^4$.
8. $(-1)^2(-1)^3$.
9. $(-2)^4(-2)^2$.
10. $(-a)^2(-a)^6$.
11. $x^n \times x^n$.
12. $t^{n+1} \times t$.
13. $P^{m+3} \times P^{m-3}$.
14. $a^9 \div a^2$.
15. $y^{12} \div y^8$.
16. $D^9 \div D$.
17. $\pi r^2 \div \pi r$.
18. $mv^n \div v^{n-2}$.
19. $at^2 \div \frac{1}{2} at$.
20. $2^8 \div 2^5$.
21. $3^{12} \div 3^{10}$.
22. $(-1)^5 \div (-1)^3$.
23. $(-n)^9 \div (-n)^5$.
24. $(x^4)^5$.
25. $(N^6)^3$.
26. $(s^{12})^4$.
27. $(k^{20})^5$.
28. $(2^3)^2$.
29. $(3^2)^4$.
30. $[(-1)^3]^6$.
31. $[(-b)^5]^4$.
32. $(x^{n-1})^n$.
33. $(mn)^4$.
34. $(a^2b^3)^6$.
35. $(x^5y)^3$.
36. $(v^9t^4)^4$.
37. $(2 \times 5)^3$.
38. $(7 \times 3)^2$.
39. $(a^nb^{2n})^5$.
40. $(\frac{3}{4})^3$.
41. $(\frac{2}{3})^4$.
42. $(\frac{4}{5})^4$.
43. $(-\frac{1}{2})^6$.
44. $(-\frac{2}{3})^3$.
45. $\left(\frac{a}{b}\right)^4$.
46. $\left(\frac{2m^2}{3n^5}\right)^3$.
47. $\left(\frac{v^3t^5}{2a^4s^2}\right)^5$.
48. $\sqrt[4]{a^{12}}$.
49. $\sqrt[6]{n^{18}}$.
50. $\sqrt{x^4y^8}$.
51. $\sqrt[5]{32r^{20}}$.
52. $\sqrt[n]{a^{2n}b^{3n}}$.
53. $(\sqrt[3]{a})^6$.
54. $(\sqrt[4]{t})^8$.
55. $(\sqrt[3]{m^2})^9$.
56. $(2a^2\sqrt[5]{b^2})^{10}$.

138. Zero Exponents. — The application of Law II, § 137, when the exponents of the dividend and divisor are equal, gives rise to a **zero exponent.**

Thus, $a^4 \div a^4 = a^{4-4} = a^0$; $r^9 \div r^9 = r^{9-9} = r^0$.

The meaning of a zero exponent may be shown as follows:

In general, $a^n \div a^n = a^{n-n} = a^0$.

But, $a^n \div a^n = 1$.

Hence, since a^0 and 1 are but two different ways of writing the same quotient,

$$a^0 = 1.$$

That is, *any base with the exponent* 0 *equals* 1.

EXERCISES

Give the values of:

1. 2^0.
2. 0.1^0.
3. 100^0.
4. $(\frac{1}{4})^0$.
5. $(-1)^0$.
6. $3^4 \times 3^0$.
7. $n^5 \div n^0$.
8. $(a^4)^0$.

139. Negative Exponents. — The application of Law II, § 137, when the exponent of the divisor is greater than that of the dividend, gives rise to a **negative exponent.**

Thus, $a^3 \div a^8 = a^{3-8} = a^{-5}$; $v^4 \div v^{10} = v^{-6}$.

It is evident that a negative exponent has not the same meaning as an exponent that is a positive whole number. For example, to say that n^{-4} indicates that n is to be taken as a factor -4 times is absurd. But this new kind of number symbol has a definite meaning which may be shown as follows:

Assuming that Law I, § 137, applies to negative exponents,

$$a^n \times a^{-n} = a^{n-n} = a^0,$$

or, $a^n \times a^{-n} = 1$.

Dividing both members of this equation by a^n,

$$a^{-n} = \frac{1}{a^n}.$$

That is, *a negative exponent indicates a power of the base that is to be used as a divisor.*

It follows from the meaning of a negative exponent that

Any factor may be changed from the numerator to the denominator of a fraction, or vice versa, by changing the sign of its exponent.

Thus,
$$\frac{a^2b^{-2}}{c^3d^{-3}} = \frac{a^2 \times \frac{1}{b^2}}{c^3 \times \frac{1}{d^3}} = \frac{\frac{a^2}{b^2}}{\frac{c^3}{d^3}} = \frac{a^2d^3}{c^3b^2}.$$

Evidently, this result might have been obtained in a single step by putting the b into the denominator and the d into the numerator and changing the signs of their exponents.

Similarly,
$$\frac{3^{-4}}{2^{-5}} = \frac{2^5}{3^4} = \frac{32}{81}.$$

EXERCISES

Find the values of:

1. 2^{-2}.
2. 2^{-5}.
3. 3^{-4}.
4. 7^{-1}.
5. 10^{-5}.
6. 1^{-1}.
7. 0.5^{-2}.
8. 0.01^{-3}.
9. $(\frac{1}{2})^{-2}$.
10. $(\frac{2}{3})^{-1}$.
11. $(\frac{1}{5})^{-3}$.
12. $(\frac{3}{8})^{-2}$.
13. $(\frac{2}{5})^{-4}$.
14. $2^{-5} \times (\frac{1}{2})^{-1}$.
15. $10^{-3} \div (\frac{3}{4})^{-2}$.
16. $(\frac{1}{2})^{-6} \times (\frac{2}{3})^{-4}$.

Write with positive exponents:

17. n^{-4}.
18. a^2b^{-3}.
19. $\pi R^2 r^{-2}$.
20. $\frac{m^{-2}}{n^{-4}}$.
21. $\frac{a^5}{b^{-5}}$.
22. $\frac{a^{-1}b^{-2}}{c^3}$.
23. $\frac{2x^{-2}}{-y^2}$.
24. $\frac{-5P^{-4}}{-3N^{-3}}$.
25. $\frac{x^{-1}y^{-2}z^{-3}}{xy^2z^3}$.
26. $\frac{x^3y^{-3}}{1}$.
27. $\frac{a^{-6}b}{-6}$.
28. $\frac{V^{-2}}{5}$.
29. $\frac{m}{n} + \frac{m^{-1}}{n^{-1}}$.
30. $\frac{a^{-1}+b^{-1}}{a^{-1}-b^{-1}}$.
31. $\frac{x^2-y^{-2}}{x^2+y^{-2}}$.
32. $\frac{(a+b)^{-1}}{(a-b)^{-1}}$.
33. $(N^{-2})^{-3}$.

Write in integral form:

34. $\dfrac{a^2}{b^2}$.

35. $\dfrac{2x}{3y}$.

36. $\dfrac{1}{-5n}$.

37. $\dfrac{v^{-2}}{t^{-2}}$.

38. $\dfrac{1}{x}+\dfrac{1}{y}$.

39. $\dfrac{a}{b}+\dfrac{b}{a}$.

40. $\dfrac{m}{n^{-1}}+\dfrac{n}{m^{-1}}$.

41. $\dfrac{1}{a^2+ab+b^2}$.

140. Fractional Exponents. — The application of Law VI, § 137, when the exponent of the base is not divisible without remainder by the index of the root, gives rise to a **fractional exponent.**

Thus, $\sqrt[5]{a^2}=a^{\frac{2}{5}}$; $\sqrt[3]{n^4}=n^{\frac{4}{3}}$; $\sqrt{5}=5^{\frac{1}{2}}$.

It is clear that a fractional exponent has not the same meaning as an integral exponent. For example, $a^{\frac{3}{5}}$ cannot indicate that a is to be used as a factor $\frac{3}{5}$ of a time. Such a statement is absurd. The meaning of every fractional exponent may be shown as follows:

Assuming that Law I, § 137, applies to fractional exponents,

$$\begin{aligned}(a^{\frac{n}{m}})^m &= a^{\frac{n}{m}}\times a^{\frac{n}{m}}\times a^{\frac{n}{m}}\cdots \text{to } m \text{ factors}\\ &= a^{\frac{n}{m}+\frac{n}{m}+\frac{n}{m}+\cdots \text{to } m \text{ terms}}\\ &= a^n.\end{aligned}$$

Hence, by definition of a root,

$$a^{\frac{n}{m}}=\sqrt[m]{a^n}.$$

That is, *a fractional exponent indicates a root of a power of the base, the numerator indicating the power of the base, and the denominator indicating the root.*

For example, $8^{\frac{2}{3}}=\sqrt[3]{8^2}=\sqrt[3]{64}=4$; $(9n^4)^{\frac{1}{2}}=\sqrt{9n^4}=3n^2$.

An expression involving negative fractional exponents may be written with positive integral exponents by first interpreting the negative signs, then the fractional forms, of the exponents.

For example, $n^{-\frac{3}{4}}=\dfrac{1}{n^{\frac{3}{4}}}=\dfrac{1}{\sqrt[4]{n^3}}$.

And $\left(\dfrac{8}{27}\right)^{-\frac{2}{3}}=\dfrac{1}{\sqrt[3]{(\frac{8}{27})^2}}=\dfrac{1}{\frac{4}{9}}=\dfrac{9}{4}$.

EXERCISES

Find the values of:

1. $4^{\frac{1}{2}}$. 6. $4^{\frac{3}{2}}$. 11. $9^{\frac{5}{2}}$. 16. $256^{-\frac{3}{4}}$.

2. $9^{\frac{1}{2}}$. 7. $81^{\frac{3}{4}}$. 12. $64^{\frac{4}{3}}$. 17. $(\frac{8}{27})^{-\frac{2}{3}}$.

3. $27^{\frac{2}{3}}$ 8. $125^{\frac{2}{3}}$. 13. $4^{-\frac{1}{2}}$. 18. $(\frac{64}{125})^{\frac{4}{3}}$.

4. $16^{\frac{3}{4}}$. 9. $32^{\frac{3}{5}}$. 14. $16^{-\frac{3}{4}}$. 19. $(\frac{81}{256})^{-\frac{1}{2}}$.

5. $25^{\frac{3}{2}}$. 10. $64^{\frac{5}{6}}$. 15. $27^{-\frac{2}{3}}$. 20. $(\frac{16}{81})^{-\frac{5}{4}}$.

21. $8^{\frac{2}{3}} \times 9^{-\frac{3}{2}}$. 22. $125^{-\frac{5}{3}} \times 32^{\frac{3}{5}}$. 23. $25^{-\frac{1}{2}} \div 81^{-\frac{1}{2}}$.

24. $(\frac{1}{27})^{-\frac{1}{3}} \div (\frac{1}{16})^{\frac{3}{4}}$. 25. $8^{-\frac{2}{3}}(25^{\frac{1}{2}} - 81^{\frac{1}{4}})$.

Express with positive integral exponents:

26. $a^{\frac{2}{3}}b^{\frac{1}{3}}$. 31. $2m^{\frac{3}{2}}n^{-\frac{2}{3}}$. 35. $\dfrac{(a-b)^{-\frac{3}{4}}}{(a+b)^{-\frac{3}{4}}}$.

27. $p^{-\frac{1}{2}}q^{\frac{1}{3}}$. 32. $9V^{-\frac{4}{5}}t^{\frac{2}{5}}$.

28. $M^{-\frac{2}{3}}V^{-\frac{3}{4}}$. 33. $\dfrac{x^{-\frac{1}{2}}y^{-\frac{1}{2}}}{z^{-1}}$. 36. $\dfrac{(h-k)^{\frac{1}{3}}}{(h-k)^{-\frac{2}{3}}}$.

29. $4x^{-\frac{5}{2}}$.

30. $-9t^{\frac{3}{4}}$. 34. $\dfrac{m^{-2}}{p^{-\frac{2}{3}}q^{-\frac{3}{2}}}$. 37. $\dfrac{m^{\frac{1}{2}}+n^{\frac{1}{2}}}{m^{-\frac{1}{2}}+n^{-\frac{1}{2}}}$.

Express without radical signs:

38. $\sqrt[5]{v^2}$. 42. $\sqrt[4]{a^2b^3mn}$. 46. $\sqrt[2n]{x^{mn}}$.

39. $\sqrt[3]{n^7}$. 43. $\sqrt[6]{27x^3y^6}$. 47. $\sqrt{a-b}$.

40. $a\sqrt{a}$. 44. $\sqrt[4]{b^3}\sqrt{c}$. 48. $\sqrt[3]{1+x^2}$.

41. $\sqrt{w}\sqrt[3]{t}$. 45. $\sqrt[mn]{a^n}$. 49. $\sqrt[m]{x^n}\sqrt[n]{x^m}$.

Write as integral expressions without radical signs:

50. $\dfrac{1}{\sqrt{1-x}}$. 51. $\dfrac{1}{\sqrt[3]{(1+x^2)^2}}$. 52. $\dfrac{\sqrt{(a+b)^3}}{\sqrt{(a-b)^{-3}}}$.

141. Laws Applied to Negative and Fractional Exponents. — In determining the meanings of negative and fractional exponents, it was assumed that Law I of positive integral exponents, § 137, is applicable to these new kinds of exponents also. It may be proved that all of the fundamental laws of positive integral exponents given in § 137 hold also for negative and fractional exponents. The proofs are omitted from this course.

Example 1. — Simplify $n^{\frac{2}{3}} \times n^{\frac{3}{4}} \times n^{\frac{1}{2}}$.

$$n^{\frac{2}{3}} \times n^{\frac{3}{4}} \times n^{\frac{1}{2}} = n^{\frac{8}{12}} \times n^{\frac{9}{12}} \times n^{\frac{6}{12}} = n^{\frac{8}{12}+\frac{9}{12}+\frac{6}{12}} = n^{\frac{23}{12}}.$$

Example 2. — Simplify $a^{-\frac{2}{5}} \div a^{-\frac{3}{10}}$.

$$a^{-\frac{2}{5}} \div a^{-\frac{3}{10}} = a^{-\frac{4}{10}} \div a^{-\frac{3}{10}} = a^{-\frac{1}{10}}.$$

Example 3. — Simplify $(A^{-4}B^{-\frac{3}{4}})^{-\frac{1}{2}}$.

$$(A^{-4}B^{-\frac{3}{4}})^{-\frac{1}{2}} = A^2B^{\frac{3}{8}}.$$

Example 4. — Simplify $(m^{\frac{1}{3}} + n^{-\frac{1}{3}})(m^{\frac{2}{3}} - m^{\frac{1}{3}}n^{-\frac{1}{3}} + n^{-\frac{2}{3}})$.

$$\begin{array}{llll}
m^{\frac{2}{3}} & - m^{\frac{1}{3}}n^{-\frac{1}{3}} & + n^{-\frac{2}{3}} & \\
m^{\frac{1}{3}} & + n^{-\frac{1}{3}} & & \\
\hline
m & - m^{\frac{2}{3}}n^{-\frac{1}{3}} & + m^{\frac{1}{3}}n^{-\frac{2}{3}} & \\
 & m^{\frac{2}{3}}n^{-\frac{1}{3}} & - m^{\frac{1}{3}}n^{-\frac{2}{3}} & + n^{-1} \\
\hline
m & & & + n^{-1}
\end{array}$$

Operations with expressions involving radicals may be performed by first expressing the radicals by means of fractional exponents.

Example 5. — Simplify $\dfrac{x-y}{\sqrt[3]{x}-\sqrt[3]{y}}$.

Expressing the radicals by means of fractional exponents, and dividing,

$$\begin{array}{r|l}
 & x^{\frac{2}{3}} + x^{\frac{1}{3}}y^{\frac{1}{3}} + y^{\frac{2}{3}} \\
\cline{2-2}
x^{\frac{1}{3}} - y^{\frac{1}{3}} & x \qquad\qquad\qquad - y \\
 & \underline{x - x^{\frac{2}{3}}y \qquad\qquad} \\
 & \qquad x^{\frac{2}{3}}y \qquad\quad - y \\
 & \qquad \underline{x^{\frac{2}{3}}y - x^{\frac{1}{3}}y^{\frac{2}{3}}} \\
 & \qquad\qquad x^{\frac{1}{3}}y^{\frac{2}{3}} - y \\
 & \qquad\qquad \underline{x^{\frac{1}{3}}y^{\frac{2}{3}} - y}
\end{array}$$

Hence, $$\frac{x-y}{\sqrt[3]{x}-\sqrt[3]{y}} = x^{\frac{2}{3}} + x^{\frac{1}{3}}y^{\frac{1}{3}} + y^{\frac{2}{3}} = \sqrt[3]{x^2} + \sqrt[3]{xy} + \sqrt[3]{y^2}.$$

EXERCISES

Simplify:

1. $A^{-3} \times A^{-2} \times A^{7}$.
2. $4\,n^{-5} \times n^{3} \times n^{3}$.
3. $R^{-1} \times R^{6} \times R^{-5}$.
4. $12\,t^{10} \times t^{-4} \times t^{-2}$.
5. $a^{\frac{1}{2}} \times a^{-\frac{3}{2}} \times a^{2}$.
6. $5\,p^{-\frac{2}{3}} \times 3\,p^{\frac{1}{2}}$.
7. $h^{-\frac{4}{5}}k^{\frac{2}{3}} \times hk^{-\frac{1}{2}}$.
8. $V^{\frac{1}{4}}t^{\frac{1}{5}} \times V^{-\frac{1}{3}}t^{-\frac{1}{4}}$.
9. $y^{2}z^{-\frac{1}{6}} \times y^{-\frac{3}{4}}z^{\frac{3}{4}}$.
10. $x^{-2} \div x^{-4}$.
11. $an^{-1} \div a^{3}n^{4}$.
12. $w^{2}v^{-3} \div w^{-1}v^{-1}$.
13. $p^{-1}q^{-2}r^{-3} \div p^{3}q^{2}r$.
14. $N^{-\frac{4}{5}} \div N^{-\frac{7}{5}}$.
15. $K^{6} \div K^{\frac{11}{2}}$.
16. $b^{\frac{7}{10}} \div a^{-1}b^{-\frac{3}{10}}$.
17. $x^{\frac{1}{2}}y^{\frac{1}{3}}z^{-\frac{1}{4}} \div xy^{-\frac{2}{3}}z$.
18. $6\,P^{-\frac{3}{2}}V^{\frac{6}{7}} \div 3\,P^{-\frac{1}{4}}V$.
19. $(a^{-2}b^{-3})^{-2}$.
20. $(x^{4}y^{-1})^{-5}$.
21. $(\pi R^{2})^{-4}$.
22. $(m^{-\frac{1}{2}}n^{-\frac{3}{2}})^{2}$.
23. $(2\,H^{\frac{3}{4}}n^{-1})^{\frac{2}{3}}$.
24. $(5\,M^{-\frac{1}{6}}A^{\frac{1}{3}})^{-3}$.
25. $(4\,x^{-\frac{2}{3}}y^{-\frac{1}{4}}z^{-1})^{-\frac{3}{2}}$.
26. $(81\,a^{12}b^{-6}c^{-9})^{-\frac{1}{12}}$.
27. $(-27\,p^{-9}d^{-\frac{3}{4}})^{-\frac{2}{3}}$.
28. $(32\,r^{-\frac{5}{3}}s^{\frac{10}{11}})^{-\frac{3}{5}}$.
29. $\sqrt{a} \times \sqrt[3]{a}$.
30. $\sqrt[4]{t^{3}} \times \sqrt[3]{t^{4}}$.
31. $\sqrt{n} \times \sqrt[4]{n} \times \sqrt{n}$.
32. $\sqrt[3]{k^{-5}} \div \sqrt[3]{k^{-8}}$.
33. $\sqrt[5]{a^{-2}x^{-1}} \div \sqrt[5]{a^{-12}x^{-6}}$.
34. $\sqrt{a^{3}} \div \sqrt[3]{a^{2}}$.
35. $\sqrt[3]{wv} \div \sqrt[4]{w^{3}v^{2}}$.
36. $\dfrac{\sqrt[3]{n^{2}}}{\sqrt[4]{mn^{3}}} \div \dfrac{\sqrt[6]{m^{3}n^{2}}}{\sqrt[12]{m^{7}n^{9}}}$.
37. $\dfrac{1}{\sqrt[3]{k^{2}n}} \div \sqrt{kn^{2}}$.
38. $\sqrt[3]{\dfrac{a^{-1}}{b^{-3}}} \div \left(\dfrac{a^{-\frac{2}{3}}b^{\frac{1}{2}}}{b^{-1}a}\right)^{-2}$.
39. $\sqrt[13]{(\sqrt[3]{t^{2}} \times \sqrt{t^{4}})^{-14}}$.
40. $\sqrt[4]{\sqrt[3]{n^{8}}}$.
41. $\sqrt[3]{\sqrt{\sqrt[4]{x^{-6}}}}$.
42. $(x^{\frac{1}{2}} + y^{\frac{1}{2}})(x^{\frac{1}{2}} - y^{\frac{1}{2}})$.
43. $(W^{\frac{3}{2}} - V^{-\frac{3}{2}})(W^{\frac{3}{2}} + V^{-\frac{3}{2}})$.

44. $(m-n) \div (m^{\frac{1}{2}}+n^{\frac{1}{2}})$.

45. $(R+r) \div (R^{\frac{1}{3}}+r^{\frac{1}{3}})$.

46. $(x^2-y^2) \div (x^{\frac{2}{3}}-y^{\frac{2}{3}})$.

47. $(x^{\frac{1}{2}}+2)(x^{\frac{1}{2}}-1)$.

48. $(3\,m^{-\frac{2}{3}}-4\,n^{\frac{5}{2}})(3\,m^{-\frac{2}{3}}+4\,n^{\frac{5}{2}})$.

49. $(a-4\,b) \div (a^{\frac{1}{2}}-2\,b^{\frac{1}{2}})$.

50. $(t^{-3}-1) \div (t^{-1}-1)$.

51. $(1+y^{-1}) \div (1+y^{-\frac{1}{3}})$.

52. $(n^{-3}+27) \div (n^{-1}+3)$.

53. $(k^{-4}-1) \div (k^{-1}-1)$.

54. $(v^2+t^2) \div (v^{\frac{2}{3}}+t^{\frac{2}{3}})$.

55. $(M^{\frac{2}{3}}+N^{-\frac{3}{4}})^2$.

56. $(a^{\frac{1}{2}}-b^{-\frac{1}{2}})^2$.

57. $(1-x^{-\frac{1}{4}})^2$.

58. $(E^{-\frac{1}{4}}+F^{-\frac{1}{4}})^2$.

59. $(e^x+e^{-x}-2)^2$.

60. $(1-x^{-1}+x^{-3})^2$.

61. $(n^{\frac{1}{4}}+1+n^{-\frac{1}{4}})^2$.

62. $(h-k) \div (\sqrt[4]{h}-\sqrt[4]{k})$.

63. $(A^{-\frac{1}{2}n}-1)^2$.

64. $(a^{\frac{2}{3}}+a^{\frac{1}{3}}b^{\frac{1}{3}}+b^{\frac{2}{3}})(a^{\frac{1}{3}}-b^{\frac{1}{3}})$.

65. $(x^{\frac{4}{3}}+x^{\frac{2}{3}}y^{\frac{2}{3}}+y^{\frac{4}{3}})(x^{\frac{4}{3}}-x^{\frac{2}{3}}y^{\frac{2}{3}}+y^{\frac{4}{3}})$.

66. $(t^{\frac{5}{4}}-t^{\frac{3}{4}}+t^{\frac{1}{4}}-t^{-\frac{1}{4}})(t^{\frac{3}{4}}+t^{\frac{1}{4}})$.

67. $(8\,n^{-2}-8\,n^2+5\,n^6-3\,n^{-6}) \div (5\,n^2-3\,n^{-2})$.

68. $(p+32\,q) \div (p^{\frac{1}{5}}+2\,q^{\frac{1}{5}})+2\,p^{\frac{3}{5}}q^{\frac{1}{5}}+8\,p^{\frac{1}{5}}q^{\frac{3}{5}}$.

69. $(\sqrt[3]{m^4}+\sqrt[3]{m^2n^2}+\sqrt[3]{n^4})(\sqrt[3]{m^2}-\sqrt[3]{n^2})$.

70. $(\sqrt[3]{E^2}-\sqrt[3]{EF}+\sqrt[3]{F^2})(\sqrt[3]{E}+\sqrt[3]{F})$.

71. $(\sqrt{x}+\sqrt[4]{x}+1)\left(\dfrac{1}{\sqrt{x}}-\dfrac{1}{\sqrt[4]{x}}+1\right)$.

72. $\left(\dfrac{4}{\sqrt[3]{n^4}}-\dfrac{12}{\sqrt[3]{n^2}}+9\right) \div \left(\dfrac{2}{\sqrt[3]{n^2}}-3\right)$.

SUPPLEMENTARY EXERCISES

Simplify:

1. $(a^{-3n}-1) \div (a^{-n}-1)$.

2. $(x^{\frac{2n}{3}}-x^{\frac{n}{3}}y^{\frac{n}{3}}+y^{\frac{2n}{3}})(x^{\frac{n}{3}}+y^{\frac{n}{3}})$.

3. $(t^{-2n}-t^{-n}+1-t^n+t^{2n})^2$.

4. $(m+n+2\sqrt[3]{mn^2}+2\sqrt[3]{m^2n}) \div (\sqrt[3]{m^2}+\sqrt[3]{n^2}+\sqrt[3]{mn})$.

Factor:

5. $16 - n^{-2}$.

6. $R^2 + 2 + R^{-2}$.

7. $27 - N^{-3}$.

8. $k^{-2} - k^{-1} - 6$.

Find the square root of:

9. $9x - 12x^{\frac{1}{2}} + 10 - 4x^{-\frac{1}{2}} + x^{-1}$.

10. $m^2 - 6m^{\frac{3}{2}}n^{\frac{1}{3}} + 15mn^{\frac{2}{3}} - 18m^{\frac{1}{2}}n + 9n^{\frac{4}{3}}$.

11. $A^2 + 8A + 12 - 16A^{-1} + 4A^{-2}$.

12. $2 + h^2k^{-2} + 2hk + k^4 + 2h^{-1}k^3 + h^{-2}k^2$.

13. $r^{\frac{4}{5}} + 4r^{\frac{3}{5}} + 4r^{\frac{2}{5}} + 2r^{\frac{1}{5}} + 4 + r^{-\frac{2}{5}}$.

14. $a^{-\frac{4}{3}} - 2a^{-1}b^{\frac{1}{3}} - 2a^{-\frac{1}{3}}b + 3a^{-\frac{2}{3}}b^{\frac{2}{3}} + b^{\frac{4}{3}}$.

CHAPTER XIV

THE BINOMIAL THEOREM

142. The Binomial Theorem. — By actual multiplication it may be shown that:

$$(a+b)^2 = a^2 + 2\,ab + b^2;$$

$$(a+b)^3 = a^3 + 3\,a^2b + 3\,ab^2 + b^3;$$

$$(a+b)^4 = a^4 + 4\,a^3b + 6\,a^2b^2 + 4\,ab^3 + b^4;$$

$$(a+b)^5 = a^5 + 5\,a^4b + 10\,a^3b^2 + 10\,a^2b^3 + 5\,ab^4 + b^5;$$

$$(a+b)^6 = a^6 + 6\,a^5b + 15\,a^4b^2 + 20\,a^3b^3 + 15\,a^2b^4 + 6\,ab^5 + b^6; \text{ etc.}$$

If $a-b$ is taken instead of $a+b$, the even-numbered terms in each of the above expansions will be negative and the odd-numbered terms will be positive. Thus,

$$(a-b)^5 = a^5 - 5\,a^4b + 10\,a^3b^2 - 10\,a^2b^3 + 5\,ab^4 - b^5.$$

A comparison of these few values of different powers of $a+b$ and $a-b$ will show that each of them may be written down, without performing the multiplication, by use of the following laws:

(1) *The first term in each case is* **a** *with an exponent equal to that of the binomial, and the last term is* **b** *with the same exponent.*

(2) *The exponents of* **a** *decrease by* 1 *in each term after the first; and* **b** *appears to the first power in the second term, and its exponents increase by* 1 *in each term after the second. The sum of the exponents of* **a** *and* **b** *in every term is equal to the exponent of the binomial.*

(3) *The numerical coefficient in the second term equals the exponent of the first term. And the coefficient of any term after the second may be obtained from the preceding term by multiplying the coeffi-*

cient of that term by the exponent of a *and dividing the product by* 1 *more than the exponent of* b *in that term.*

(4) *If* a *and* b *are both positive, the signs of all terms are positive; if* b *is negative, the signs of all even-numbered terms are negative and all odd-numbered terms positive.*

(5) *The number of terms is greater by* 1 *than the exponent of the binomial.*

These laws taken collectively, by means of which any power of any binomial may be written down directly, constitute what is known as the **Binomial Theorem.***

NOTE. — If the coefficients of each of the various powers of $a+b$, including the zero and first powers, are written down in order in a row, these rows of numbers form a triangular arrangement, as shown in the margin. This arrangement is known as *Pascal's Triangle.* Any number may be obtained by adding the two numbers, one to the right and the other to the left of it, in the row above. The triangle may thus be continued indefinitely.

1
1 1
1 2 1
1 3 3 1
1 4 6 4 1
• • • • • •

In writing down the expansion of any power of a binomial by the Binomial Theorem, if the terms of the binomial contain coefficients or exponents, it is wise to inclose each of these terms in parentheses, and then to simplify in a second step the expression obtained. In this way confusion in finding the exponents and coefficients of the various terms of the expansion will be avoided.

EXAMPLE. — Find by the Binomial Theorem $(2x^2 - 3y^3)^4$.

$$(2x^2 - 3y^3)^4 = (2x^2)^4 - 4(2x^2)^3(3y^3) + 6(2x^2)^2(3y^3)^2 - 4(2x^2)(3y^3)^3$$
$$+ (3y^3)^4 = 16x^8 - 96x^6y^3 + 216x^4y^6 - 216x^2y^9 + 81y^{12}.$$

---

* The Binomial Theorem has a long and interesting history. The ancient Hindus were familiar with special cases of the theorem, and used it in extracting square roots and cube roots. Sir Isaac Newton, the great mathematician and scientist, was the first to recognize the general truth of the theorem, about the year 1665; and he used it in making some of his greatest mathematical discoveries. Napier was led by use of the Binomial Theorem to the discovery of logarithms. (See Chapter XV.)

EXERCISES

Find by the Binomial Theorem:

1. $(a+b)^7$.
2. $(a-b)^7$.
3. $(m+n)^8$.
4. $(m-n)^8$.
5. $(x+y)^9$.
6. $(v-t)^{10}$.
7. $(h+k)^{12}$.
8. $(P-Q)^{13}$.
9. $(1+x)^4$.
10. $(1-n)^5$.
11. $(a+1)^6$.
12. $(2x+y)^5$.
13. $(a^2-b^2)^8$.
14. $(M-3N^2)^6$.
15. $(2V+3t)^6$.
16. $(1-2x^3)^4$.
17. $(2p^2+q)^7$.
18. $(1-3V^3)^5$.
19. $(y^3+z^3)^{10}$.
20. $(1+2b^4)^8$.
21. $(1-\frac{1}{2}n)^6$.
22. $(\frac{1}{2}E^2+\frac{1}{8}F^2)^6$.
23. $(\frac{2}{3}a^4-1)^5$.
24. $(6R^2+\frac{1}{2})^4$.

Find the first four terms of:

25. $(a+b)^{18}$.
26. $(a-b)^{20}$.
27. $(1+N)^{40}$.
28. $(t-1)^{30}$.
29. $(x+2)^{20}$.
30. $(M^2-2N)^{24}$.
31. $(w^2-3v^2)^{22}$.
32. $(\frac{1}{2}x+2y)^{30}$.
33. $\left(n-\frac{2}{n}\right)^{40}$.
34. $\left(\frac{2a}{b}+\frac{b^2}{4a^2}\right)^{16}$.
35. Write the first five terms of $(a-b)^n$.

Compute the following correct to two decimal places:

36. $(1.1)^{12}$.

SUGGESTION. — $(1.1)^{12} = (1+.1)^{12} =$ etc.

37. $(1.2)^{10}$.
38. $(1.04)^8$.
39. $(.31)^{10}$.
40. $(2.5)^{12}$.
41. Form the first ten lines of Pascal's Triangle.

143. Proof for Positive Integral Exponents. — That the Binomial Theorem holds true for all positive integral powers of any binomial may be proved as follows:

Assuming that the theorem holds true for the nth power of $a+b$, where n is any positive integer, we may write $(a+b)^n =$

$$a^n + na^{n-1}b + \frac{n(n-1)}{1\cdot 2}a^{n-2}b^2 + \frac{n(n-1)(n-2)}{1\cdot 2\cdot 3}a^{n-3}b^3 + \cdots + b^n.$$

Multiplying both members of this equation by $a+b$,

$$(a+b)^{n+1}=a^{n+1}+(n+1)a^nb+\frac{(n+1)(n)}{1\cdot 2}a^{n-1}b^2$$
$$+\frac{(n+1)(n)(n-1)}{1\cdot 2\cdot 3}a^{n-2}b^3+\cdots+b^{n+1}.$$

Now it is seen that this value of $(a+b)^{n+1}$ is of the same form as the value of $(a+b)^n$, and that the result is the same as if it had been obtained directly by use of the Binomial Theorem.

Hence, if the theorem holds true for the nth power of $a+b$, it holds true for the $(n+1)$th power.

But it is proved by actual multiplication that the theorem holds true for the sixth power. Hence, since it holds for the sixth power, it holds for the seventh power. Now since it holds for the seventh power, it holds also for the eighth power; since it holds for the eighth power, it holds also for the ninth power; etc.

Hence, it holds true for all positive integral powers of $a+b$.

By similar proof it may be shown to hold for all positive integral powers of $a-b$.

NOTE. — It is unsafe to reason that just because a law holds true for a few, or even many special cases, it always holds true. Certain laws have been shown to hold true for many special cases, yet finally fail. Thus, the ancient Chinese thought that if $2^{n-1}-1$ were exactly divisible by n, then n must be a prime number. This holds true for all values of n up to 341. But for this value it fails. Hence the necessity is seen of finding a general proof of every law, such as that in this section. Mathematics undertakes to find general proofs of all of its principles.

144. Negative or Fractional Powers of Binomials. — The Binomial Theorem holds true also for negative and fractional powers of a binomial $a+b$, provided that the absolute value of a is greater than that of b, except that in these cases the number of terms in the expansion is unlimited. The proof is too difficult for this course.

EXAMPLE 1. — Expand to 5 terms $(1+x^2)^{-2}$.

$$(1+x^2)^{-2}=(1)^{-2}+(-2)(1)^{-3}(x^2)+(3)(1)^{-4}(x^2)^2+(-4)(1)^{-5}(x^2)^3$$
$$+(5)(1)^{-6}(x^2)^4+\cdots$$
$$=1-2x^2+3x^4-4x^6+5x^8-\cdots,$$

Example 2. — Expand to 5 terms $(1-2n)^{\frac{1}{2}}$.

$$(1-2n)^{\frac{1}{2}} = (1)^{\frac{1}{2}} - (\tfrac{1}{2})(1)^{-\frac{1}{2}}(2n) + (-\tfrac{1}{8})(1)^{-\frac{3}{2}}(2n)^2 - (\tfrac{1}{16})(1)^{-\frac{5}{2}}(2n)^3$$
$$+ (-\tfrac{5}{128})(1)^{-\frac{7}{2}}(2n)^4 - \cdots$$
$$= 1 - n - \tfrac{1}{2}n^2 - \tfrac{1}{2}n^3 - \tfrac{5}{8}n^4 - \cdots.$$

EXERCISES

Expand to 5 terms:

1. $(1+n)^{-1}$.
2. $(1-t)^{-3}$.
3. $(1+x)^{\frac{1}{2}}$.
4. $(1-R)^{\frac{1}{3}}$.
5. $(1+a)^{\frac{1}{4}}$.
6. $(1-x^3)^{-1}$.
7. $(1+k^2)^{-2}$.
8. $(1+P^2)^{-\frac{1}{2}}$.
9. $(1+4a^2)^{-\frac{1}{2}}$.
10. $(1-9y^3)^{-\frac{1}{3}}$.
11. $(a^2-b^2)^{-\frac{1}{4}}$.
12. $(x^{\frac{1}{2}}+y^{\frac{1}{2}})^{\frac{1}{2}}$.
13. $(m^{-\frac{1}{2}}-n^{-\frac{1}{2}})^{-\frac{1}{2}}$.
14. $(2v^{\frac{2}{3}}+t^{\frac{3}{2}})^{\frac{2}{3}}$.
15. $(3p^{-\frac{3}{4}}-2q^{-1})^{-\frac{1}{3}}$.

145. Extraction of Roots by the Binomial Theorem. — The Binomial Theorem may be used to extract roots of arithmetical numbers. The process is best shown by an example.

Example. — Find to 3 decimal places the value of $\sqrt[5]{29}$.

$$\sqrt[5]{29} = 29^{\frac{1}{5}} = (2^5-3)^{\frac{1}{5}}$$
$$= (2^5)^{\frac{1}{5}} - (\tfrac{1}{5})(2^5)^{-\frac{4}{5}}(3) + \frac{(\frac{1}{5})(-\frac{4}{5})}{1\cdot 2}(2^5)^{-\frac{9}{5}}(3)^2$$
$$- \frac{(\frac{1}{5})(-\frac{4}{5})(-\frac{9}{5})}{1\cdot 2\cdot 3}(2^5)^{-\frac{14}{5}}(3)^3 + \cdots$$
$$= 2 - \tfrac{3}{80} - \tfrac{9}{6400} - \tfrac{81}{1024000} - \cdots$$
$$= 2 - 0.0375 - 0.0014 - 0.0001 - \cdots$$
$$= 1.961, \text{ approximately.}$$

A similar process may be used in any case. Hence the rule:

To find a root of a number, separate the number into two parts, the first of which is the nearest possible perfect power of which the required root can be found; then expand the resulting binomial by the Binomial Theorem, and combine the values of the terms thus obtained.

Thus, $\sqrt[3]{65} = \sqrt[3]{64+1} = (4^3+1)^{\frac{1}{3}}$; $\sqrt[4]{623} = \sqrt[4]{625-2} = (5^4-2)^{\frac{1}{4}}$.

It is evident that the first few terms of the expansion will give a close approximation to the value of the root, if the successive terms decrease in value rapidly; that is, if the second term of the binomial is much smaller than the first.

NOTE.—A shorter method of finding the approximate value of any root of a number is by the use of logarithms, as shown in the next chapter.

EXERCISES

Find to 3 places of decimals the value of:

1. $\sqrt[4]{15}$.
2. $\sqrt[3]{28}$.
3. $\sqrt[3]{66}$.
4. $\sqrt[3]{25}$.
5. $\sqrt[3]{130}$.
6. $\sqrt[3]{219}$.
7. $\sqrt[3]{9}$.
8. $\sqrt[4]{261}$.
9. $\sqrt[5]{240}$.
10. $\sqrt[6]{67}$.
11. $\sqrt[5]{3120}$.
12. $\sqrt[6]{730}$.
13. $\sqrt[4]{85}$.
14. $\sqrt[5]{29}$.
15. $\sqrt[6]{70}$.

146. The *r*th Term of the Binomial Expansion. — In the expansion $(a+b)^n = a^n + na^{n-1}b + \frac{n(n-1)}{1 \cdot 2}a^{n-2}b^2 + \frac{n(n-1)(n-2)}{1 \cdot 2 \cdot 3}a^{n-3}b^3 + \cdots + b^n$, it is seen that the numerator of the coefficient of the 3rd term is the product of 2 factors, $n(n-1)$; the numerator of the coefficient of the 4th term is the product of 3 factors, $n(n-1)(n-2)$; etc. In general, the numerator of the coefficient of the rth term is the product of $r-1$ factors, $n(n-1)(n-2)\cdots$.

The denominator of the coefficient of the third term is $1 \cdot 2$; of the fourth term, $1 \cdot 2 \cdot 3$; and in general, of the rth term, $1 \cdot 2 \cdot 3 \cdot 4 \cdots$ to $r-1$ factors.

The exponent of a in the third term is $n-2$; in the fourth term $n-3$; and in general, in the rth term, $n-(r-1)$.

The exponent of b in the third term is 2; in the fourth term 3; and in general, in the rth term, $r-1$.

Hence, in the expansion of $(a+b)^n$ the

$$r\text{th term} = \frac{n(n-1)(n-2)\cdots \text{to } r-1 \text{ factors}}{1 \cdot 2 \cdot 3 \cdots \text{to } r-1 \text{ factors}}\, a^{n-(r-1)}b^{r-1}.$$

In the expansion of $(a-b)^n$, the rth term is the same expression, except that it is preceded by $+$ only when r is *odd*, and by $-$ when r is *even*.

By substituting the values of a, b, n, and r in this formula for the rth term, we may write down directly any desired term of the expansion of any power of any binomial, without writing the preceding terms.

Example. — Find the eighth term in the expansion of $(1-t^3)^{-2}$.

Here $a=1$, $b=t^3$, $n=-2$, and $r=8$.

Hence, the eighth term

$$=-\frac{(-2)(-3)(-4)(-5)(-6)(-7)(-8)}{1\cdot 2\cdot 3\cdot 4\cdot 5\cdot 6\cdot 7}(1)^{-9}(t^3)^7$$

$$=8\,t^{21}.$$

EXERCISES

Find:

1. The fourth term in the expansion of $(a^2+b^2)^6$.
2. The fifth term in the expansion of $(1+x)^{-4}$.
3. The eighth term in the expansion of $(1-t^2)^{20}$.
4. The sixth term in the expansion of $(m-n)^{40}$.
5. The fourth term in the expansion of $(x^3+y^3)^{12}$.
6. The tenth term in the expansion of $(A^4-B^2)^{-1}$.
7. The seventh term in the expansion of $(1+2x^3)^{\frac{1}{2}}$.
8. The twelfth term in the expansion of $(1-V^{-1})^{-\frac{1}{2}}$.
9. The eighth term in the expansion of $(x^{\frac{1}{2}}+y^{\frac{1}{2}})^{\frac{1}{4}}$.
10. The ninth term in the expansion of $(R^{-1}-r^2)^{-\frac{1}{3}}$.
11. The fourth term in the expansion of $(2a+b^2)^{\frac{1}{2}}$.
12. The sixteenth term in the expansion of $(1-a^{-\frac{1}{2}})^{-\frac{1}{4}}$.
13. The sixth term in the expansion of $(2-k)^{-3}$.
14. The eighth term in the expansion of $(2+\frac{1}{2}n^2)^{\frac{1}{2}}$.
15. The eleventh term in the expansion of $(x^{\frac{2}{3}}+x^{-\frac{2}{3}})^{\frac{2}{3}}$.
16. The fifth term in the expansion of $(e^{\frac{1}{2}}-e^{-\frac{1}{2}})^{-2}$.

SUPPLEMENTARY EXERCISES

Expand:

1. $(\sqrt{x}-\sqrt{y})^8$.

2. $(\sqrt[3]{a^2}+\sqrt[3]{b^2})^6$.

3. $(m\sqrt{m}+n\sqrt{n})^{10}$.

4. $(e^2\sqrt[4]{e}-1)^{12}$.

5. $\left(\frac{\sqrt{a}}{b}+\frac{\sqrt{b}}{a}\right)^{10}$.

6. $\left(\frac{3m}{n^2}+n\sqrt{m}\right)^4$.

7. $\left(\frac{\sqrt{x}}{\sqrt[3]{y^2}}-\sqrt{\frac{y}{x}}\right)^6$.

8. $\left(\frac{a\sqrt[3]{a}}{\sqrt[4]{b}}+\frac{\sqrt[5]{b}}{2a}\right)^4$.

By grouping terms, express as binomials and expand:

9. $(1+x-x^2)^4$.

SUGGESTION. — $(1+x-x^2)^4=([1+x]-x^2)^4$
$=[1+x]^4-4[1+x]^3[x^2]+\cdots$.
Now expand $[1+x]^4$, $[1+x]^3$, etc., and combine terms.

10. $(x^2+y^2+z^2)^3$.

11. $(2-M+N)^4$.

12. $(a-b+c-d)^4$.

13. $(1-t+t^2-t^3)^3$.

14. $(2a-b+3c^2)^4$.

15. $(n^{-2}+1+n^2)^4$.

16. Write the formula for the $(r+1)$th term in the expansion of $(a+b)^n$.

17. Write the formula for the $(r-1)$th term in the expansion of $(a+b)^n$.

Find:

18. The sixth term in the expansion of $(v\sqrt{w}-w\sqrt{v})^8$.

19. The fifth term in the expansion of $\left(\frac{\sqrt[5]{a^2}}{b}+\frac{\sqrt[5]{b^2}}{a}\right)^{10}$.

20. Find to four decimal places the value of $\sqrt[3]{220}$.

CHAPTER XV

LOGARITHMS

147. Logarithms. — Long and complicated computations, such as are encountered often in practical work, may be shortened and simplified by use of **logarithms.**

Thus, to compute the area of a triangle by the formula *Area* $= \sqrt{s(s-a)(s-b)(s-c)}$ one must find the product of four factors and then take the square root of this product. By the use of logarithms, as we shall see later, the multiplication is replaced by simple column addition, and the extraction of the root by division by 2.

The **logarithm** of a number is the *exponent* which indicates the power to which a given base must be raised to produce that number. That is, if $b^e = n$, then e is the logarithm of n to the base b, and is written

$$e = \log_b n.$$

Thus, since $2^3 = 8$, $\log_2 8 = 3$. Since $5^4 = 625$, $\log_5 625 = 4$. Since $4^{-2} = \frac{1}{16}$, $\log_4 \frac{1}{16} = -2$.

By the definition of a logarithm, the equations $e = \log_b n$ and $b^e = n$ express the same relation.

EXERCISES

Express the following relations in terms of logarithms:

1. $2^5 = 32$.

SOLUTION. — $\log_2 32 = 5$.

2. $2^6 = 64$.
3. $5^2 = 25$.
4. $3^4 = 81$.
5. $4^3 = 64$.
6. $5^3 = 125$.
7. $7^3 = 343$.
8. $10^4 = 10000$.
9. $5^6 = 15625$.
10. $6^{-2} = \frac{1}{36}$.
11. $4^{-3} = \frac{1}{64}$.
12. $16^{\frac{1}{4}} = 2$.
13. $8^{-\frac{2}{3}} = \frac{1}{4}$.

Express the following relations by means of exponents:

14. $\log_3 9 = 2$.

SOLUTION — $3^2 = 9$.

15. $\log_2 16 = 4$.
16. $\log_4 16 = 2$.
17. $\log_3 27 = 3$.
18. $\log_{10} 1000 = 3$.
19. $\log_6 216 = 3$.
20. $\log_4 256 = 4$.
21. $\log_8 4 = \frac{2}{3}$.
22. $\log_{27} 81 = \frac{4}{3}$.
23. $\log_7 \frac{1}{49} = -2$.
24. $\log_3 \frac{1}{243} = -5$.
25. $\log_{10} 0.001 = -3$.
26. $\log_{100} 0.001 = -\frac{3}{2}$.

Find the values of the following logarithms:

27. $\log_2 16$.
28. $\log_3 27$.
29. $\log_{10} 100$.
30. $\log_{10} 0.0001$.
31. $\log_4 64$.
32. $\log_2 0.5$.
33. $\log_2 0.25$.
34. $\log_8 \frac{1}{64}$.
35. $\log_6 216$.
36. $\log_5 \frac{1}{125}$.
37. $\log_{10} 1$.
38. $\log_4 1$.

To the base 4 what numbers have the following logarithms?

39. 1.
40. 3.
41. 5.
42. -2.
43. -4.
44. $\frac{1}{2}$.
45. $-\frac{1}{2}$.
46. $-\frac{3}{2}$.

148. Common Logarithms. — The logarithm of a number to the base 10 is called a **common logarithm**.

Thus, $\log_{10} 6$, $\log_{10} \frac{1}{16}$, $\log_{10} 0.001$ are *common logarithms.*

The logarithms of all arithmetical numbers to any one base constitute a **system of logarithms**. The common logarithms of all arithmetical numbers constitute what is called the **common system** of logarithms.*

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* The common system of logarithms was first worked out by an Englishman named Henry Briggs, in the early part of the seventeenth century. Another important system is the **Napierian** system, named after John Napier, an Englishman who was a contemporary of Briggs. Napier was the real inventor of logarithms, and Briggs got his inspiration from Napier. The base of the Napierian system is the sum of an infinite series $1 + \frac{1}{1} + \frac{1}{1 \cdot 2} + \frac{1}{1 \cdot 2 \cdot 3} + \frac{1}{1 \cdot 2 \cdot 3 \cdot 4} + \cdots$. The sum of this series is approximately 2.7182818, and is represented in mathematical writings by the letter e.

While the common system is used in practical computations, the Napierian system is used in theoretical investigations.

The common system is superior to any other system of logarithms for practical use because its base 10 is also the base of our decimal system of numbers.

The rest of this chapter will be devoted to common logarithms, and in the following sections the base will be understood to be 10 and will not be written.

149. Characteristic and Mantissa. — From the definition of a logarithm,

$$\begin{array}{llll}
& \text{since } 10^0 = 1, & \log 1 = 0; & \\
& \text{since } 10^1 = 10, & \log 10 = 1; & \\
& \text{since } 10^2 = 100, & \log 100 = 2; & \\
& \text{since } 10^3 = 1000, & \log 1000 = 3; & \text{etc.} \\
\text{And,} & \text{since } 10^{-1} = 0.1, & \log 0.1 = -1; & \\
& \text{since } 10^{-2} = 0.01, & \log 0.01 = -2; & \\
& \text{since } 10^{-3} = 0.001, & \log 0.001 = -3; & \text{etc.}
\end{array}$$

It is evident from the above that the logarithm of a positive integral power of 10 is a positive integer, and that the logarithm of a negative integral power of 10 is a negative integer.

Furthermore, since 65, say, is greater than 10^1 and less than 10^2, log 65 is greater than 1 and less than 2. Hence, $\log 65 = 1$ + *some decimal.* Likewise, since 382 is greater than 10^2 and less than 10^3, log 382 is greater than 2 and less than 3. Hence, $\log 382 = 2$ + *some decimal.* Evidently, in general, the logarithm of any positive number, except a positive or negative integral power of 10, consists of *an integral and a decimal part.*

Thus, $\log 825 = 2.916+$.

The integral part of a logarithm is called its **characteristic**, and the decimal part is called its **mantissa**.

150. Determination of the Characteristic. — A number having one figure in its integral part lies between 10^0 and 10^1. Hence, its logarithm lies between 0 and 1; *i.e.* it equals 0 + *some decimal.* A number having two figures in its integral part lies between 10^1 and 10^2. Hence, its logarithm lies between 1 and 2; *i.e.* it equals 1 + *some decimal.* Similarly, if a number has three figures in its integral part, its logarithm lies between 2 and 3; *i.e.* it equals 2 + *some decimal,* and so on. Therefore,

If a number is greater than 1, *the characteristic of its logarithm is positive, and is less by* 1 *than the number of figures in the integral part of the number.*

Thus, in log 6841.27, the characteristic is 3. In log 362.781, the characteristic is 2.

Again, a number less than 1, and having no zero immediately following the decimal point, lies between 10^0 and 10^{-1}. Hence, the logarithm lies between 0 and -1; *i.e.* it equals -1 + *some decimal.* A number less than 1, and having one zero immediately following the decimal point, lies between 10^{-1} and 10^{-2}. Hence, its logarithm lies between -1 and -2; *i.e.* it equals -2 + *some decimal.* Similarly, if a number less than 1 has two zeros immediately following the decimal point, its logarithm lies between -2 and -3; *i.e.* it equals -3 + *some decimal,* and so on. Therefore,

If a number is less than 1, *the characteristic of its logarithm is negative, and is greater by* 1 *than the number of zeros immediately following the decimal point.*

Thus, in log 0.0683 the characteristic is -2. In log 0.0008 the characteristic is -4. In log 0.3974 the characteristic is -1.

In writing the logarithm of a number the mantissa is always positive, and if the characteristic is negative, the minus sign is written above the characteristic to signify that it applies to the characteristic alone.

Thus log 0.00357 = $\bar{3}.5527$. This means $-3 + 0.5527$.

EXERCISES

Find the characteristics in the logarithms of:

1. 28.
2. 521.
3. 0.07.
4. 0.00064.
5. 6.871.
6. 98743.
7. 1.426.
8. 3.1416.
9. 0.729.
10. 5280.
11. 12.65.
12. 0.0082.

151. Tables of Mantissas. — Since changing the position of the decimal point in a number is equivalent to multiplying or dividing the number by some integral power of 10, which has the effect of changing its logarithm by some whole number only,

The common logarithms of numbers which do not differ except in the positions of the decimal point have the same mantissa.

For example, $\log 2680 = 3.4281$; $\log 268 = 2.4281$; $\log 26.8 = 1.4281$; $\log 2.68 = 0.4281$; $\log 0.268 = \bar{1}.4281$; $\log 0.0268 = \bar{2}.4281$. The mantissas are the same in all of these logarithms.

Since the mantissa of the logarithm of a number depends only upon the sequence of figures, and not upon the position of the decimal point, it is necessary to know only the mantissas of the logarithms of whole numbers. The common logarithms of sets of consecutive whole numbers have been computed and tabulated. On pages 252 and 253 is a table which contains the mantissas of the logarithms of all whole numbers from 1 to 1000. Logarithms may be computed to any number of decimal places, the number taken depending upon the degree of accuracy required in their use. In the table in this book the mantissas are computed to four decimal places. In this table the first two figures of each number are found in the column headed N, and the third figure in the horizontal line at the top of the table. The mantissas, with decimal points omitted, are found in the columns headed 0, 1, 2, 3, etc.

To find the logarithm of a number, find its characteristic by the rules in § 150, and look in the table for the mantissa. If a number has less than three figures, annex ciphers until it has three figures, in looking for the mantissa.

Thus to find the mantissa of log 38, look for the mantissa of log 380. To find the mantissa of log 3, look for the mantissa of log 300.

152. To find the Logarithm of a Given Number. — (*a*) *When the given number contains not more than three figures,* the mantissa of its logarithm is obtained directly from the table.

Example 1. — Find log 32.7.

By § 150 its characteristic is 1. The required mantissa is the mantissa of log 327.

Look for 32 in the column headed N in the table. Looking along the horizontal line of numbers opposite 32, to the column headed 7, we find .5145, the required mantissa. Hence,

$$\log 32.7 = 1.5145.$$

Example 2. — Find log 0.91.

By § 150 the characteristic is -1. The required mantissa is the mantissa of log 910. This is opposite 91 in the column headed 0, and is seen to be .9590. Hence,

$$\log 0.91 = \bar{1}.9590.$$

(*b*) *When the given number contains more than three figures*, use is made of the principle that when the difference of two numbers is small compared with either of them, the difference of the numbers is approximately proportional to the difference of their logarithms.

Example 3. — Find log 2.8465.

Shift the decimal point until it follows the third figure.

The required mantissa is that of log 284.65.

Now 284.65 is greater than 284 by .65.

The mantissa of log 284 = .4533.

The mantissa of log 285 = .4548.

Subtracting, .4548 − .4533 = .0015.

Hence, an increase of 1 in 284 causes an increase of .0015 in the corresponding mantissa. Therefore, an increase of .65 will cause an increase of .65 × .0015, or approximately .0010, in the mantissa.

Adding .0010 to the mantissa of log 284 gives .4543.

Hence, attaching the characteristic 0,

$$\log 2.8465 = 0.4543.$$

To find the mantissa of the logarithm of a number of more than three figures, set the decimal point after the third figure. Find in the table the mantissas corresponding to the integral part of the number thus pointed off and to the number 1 greater. Take the difference of these mantissas, and multiply the result by the decimal part of the number as pointed off. Add this product to the smaller mantissa. This gives the mantissa required.

EXERCISES

Find the logarithms of:

1. 215.
2. 673.
3. 940.
4. 717.
5. 4.62.
6. 19.9.
7. 830.
8. 16.
9. 29.
10. 8.5.
11. 100.
12. 900.
13. 1.
14. 2.
15. 1684.
16. 34.27.
17. 100.5.
18. 926.81.
19. 0.8632.
20. 0.00315.
21. 0.06843.
22. 4268.4.
23. 1.096.
24. 0.00012.
25. 99.99.
26. 2031.7.
27. 0.0083326.
28. 0.50416.
29. 68593.
30. 0.074803.

153. To find a Number whose Logarithm is Given. —

(*a*) *When the given mantissa is found in the table*, the sequence of figures of the required number may be obtained by reversing the process (*a*), § 152. The position of the decimal point is determined by reversing the rules in § 150.

Example. — Find the number whose logarithm is $\bar{2}.9325$.

Looking in the table, we find the mantissa .9325 opposite 85 and in the column headed 6. Hence,

$$.9325 = \text{mantissa of } \log 856.$$

Since the characteristic is − 2, the number must be less than 1, and must contain one zero immediately to the right of the decimal point. Hence,

$$\bar{2}.9325 = \log 0.0856.$$

(*b*) *When the given mantissa is not found in the table*, the required number is obtained by reversing the process of (*b*), § 152.

Example 2. — Find the number whose logarithm is 3.6496.

The mantissa .6496 is not in the table, but the mantissas of the table between which it lies in value are .6493 and .6503.

Now $.6493 =$ mantissa of log 446,

and $.6503 =$ mantissa of log 447.

Subtracting, $.6496 - .6493 = .0003$,

and $.6503 - .6493 = .0010$.

Hence, an increase of .0010 in the mantissa .6493 causes an increase of 1 in the corresponding number. Therefore, an increase of .0003 will cause an increase of $.0003 \div .0010$, or .3, in the number.

Adding .3 to 446 gives 446.3.

Therefore, $.6496 =$ mantissa of log 446.3.

Since the characteristic is 3, therefore

$$3.6496 = \log 4463.$$

To find the number corresponding to a given mantissa that is not in the table, take from the table the two mantissas between which the value of the given mantissa lies, and note the numbers corresponding to them. Subtract the smaller mantissa of the table from the larger and also from the given one. Divide the smaller remainder by the larger remainder, and add the quotient obtained to the number corresponding to the smaller mantissa of the table. This gives the sequence of figures of the required number.

EXERCISES

Find the numbers whose logarithms are:

1. 2.6739.
2. 1.7388.
3. 3.0828.
4. 0.7910.
5. $\bar{2}.8971$.
6. $\bar{3}.9777$.
7. $\bar{1}.6180$.
8. 0.2279.
9. 1.7582.
10. 3.8615.
11. 0.1847.
12. $\bar{1}.4609$.
13. $\bar{2}.6304$.
14. 5.6887.
15. 2.4783.
16. 3.6028.
17. $\bar{3}.7255$.
18. 2.8980.
19. 4.0096.
20. 1.4703.
21. $\bar{2}.9765$.
22. 0.8460.
23. $\bar{1}.4072$.
24. $\bar{5}.0220$.
25. 3.0392.
26. $\bar{4}.4756$.
27. 0.8735.
28. 1.5734.

154. General Principles of Logarithms. — Since a logarithm is an *exponent*, the following general principles of logarithms may be proved by the laws of exponents (see Chapter XIII):

I. *The logarithm of a product equals the sum of the logarithms of its factors. That is,*

$$\log_a mn = \log_a m + \log_a n.$$

For, let $a^x = m$, and $a^y = n$.

Then $mn = a^x \cdot a^y = a^{x+y}$.

Hence, $\log_a mn = x + y = \log_a m + \log_a n$.

Evidently, this principle may be extended to hold for a product of any number of factors. In general,

$$\log_a mnp \cdots = \log\ m + \log_a n + \log_a p + \cdots.$$

II. *The logarithm of a quotient equals the logarithm of the dividend minus the logarithm of the divisor. That is,*

$$\log_a \frac{m}{n} = \log_a m - \log_a n.$$

For, let $a^x = m$, and $a^y = n$.

Then $\frac{m}{n} = a^x \div a^y = a^{x-y}$.

Hence, $\log_a \frac{m}{n} = x - y = \log_a m - \log_a n$.

III. *The logarithm of a power of a number equals the logarithm of the number, multiplied by the exponent of the power. That is,*

$$\log_a n^p = p \log_a n.$$

For, let $a^x = n$.

Then $n^p = (a^x)^p = a^{px}$.

Hence, $\log_a n^p = px = p \log_a n$.

IV. *The logarithm of a root of a number equals the logarithm of the number, divided by the index of the root. That is,*

$$\log_a \sqrt[r]{n} = \frac{\log_a n}{r}.$$

For, let $a^x = n$.

Then $\sqrt[r]{n} = \sqrt[r]{a^x} = a^{\frac{x}{r}}$.

Hence, $\log_a \sqrt[r]{n} = \frac{x}{r} = \frac{\log_a n}{r}$, or $\frac{1}{r} \log_a n$.

By use of the above principles it is possible to replace the operations of multiplication and division by those of addition and subtraction, raising a number to a power by multiplication, and extraction of a root by division.

EXAMPLE 1. — Express $\log_a \frac{xy^2}{z\sqrt{w}}$ in terms of $\log_a x$, $\log_a y$, $\log_a z$, and $\log_a w$.

$$\log_a \frac{xy^2}{z\sqrt{w}} = \log_a xy^2 - \log_a z\sqrt{w} \qquad \text{(By II)}$$

$$= \log_a x + \log_a y^2 - \log_a z - \log_a \sqrt{w} \qquad \text{(By I)}$$

$$= \log_a x + 2\log_a y - \log_a z - \tfrac{1}{2}\log_a w. \qquad \text{(By III, IV)}$$

EXERCISES

Express the following in terms of $\log_a x$, $\log_a y$, $\log_a z$, and $\log_a w$:

1. $\log_a xyzw$.
2. $\log_a x^2y^2$.
3. $\log_a 2\,xy^2z^3$.
4. $\log_a \frac{x^2y}{z^3}$.
5. $\log_a \frac{5\,x}{yzw}$.
6. $\log_a \frac{8\,x^2y^2}{z^3w}$.
7. $\log_a \left(\frac{xy}{zw}\right)^2$.
8. $\log_a x^{\frac{2}{3}}y^{-\frac{3}{4}}$.
9. $\log_a x^{\frac{1}{3}}y^{\frac{1}{3}}z^{-\frac{1}{4}}$.
10. $\log_a \frac{\sqrt{x}}{\sqrt{y}}$.
11. $\log_a \frac{\sqrt{x}\sqrt{y}}{z\sqrt{w}}$.
12. $\log_a \frac{\sqrt[3]{xy^2}}{\sqrt{z}\sqrt[3]{w}}$.
13. $\log_a \frac{\sqrt[4]{zy^3}}{\sqrt[3]{x}}$.
14. $\log_a \sqrt{\frac{x}{y}} + \log_a \sqrt{\frac{z}{w}}$.
15. $\log_a \sqrt[3]{\frac{x}{y}}\sqrt[4]{\frac{y}{x}}$.

Express:

16. $\log \sqrt[3]{1728}$ in terms of $\log 1728$.
17. $\log \sqrt{2}\sqrt[3]{9}$ in terms of $\log 2$ and $\log 9$.
18. $\log \frac{\sqrt{98}}{\sqrt[3]{56}}$ in terms of $\log 98$ and $\log 56$.

155. Computation by Logarithms. — By use of the principles in § 154 and the processes of § 153 computations otherwise long and tedious may be performed with ease.

Example. — Find the value of $\sqrt[3]{\frac{621.3 \times 3.247}{71.8}}$.

Taking the logarithm,

$$\log \sqrt[3]{\frac{621.3 \times 3.247}{71.8}} = \tfrac{1}{3}(\log 621.3 + \log 3.247 - \log 71.8).$$

$$\begin{array}{rl} \log 621.3 = & 2.7933 \\ \log 3.247 = & \underline{0.5115} \\ & 3.3048 \\ \log 71.8 = & \underline{1.8561} \\ & 3\overline{)1.4487} \\ & 0.4829 \end{array}$$

Now, $0.4829 = \log 3.04.$

Hence, $\sqrt[3]{\frac{621.3 \times 3.247}{71.8}} = 3.04$, approximately.

In using a logarithm with a negative characteristic, it is convenient usually to add some number to the logarithm, in order to make the characteristic positive, then to indicate the subtraction of the number added.

Example. — Find the fifth root of 0.00327.

Taking the logarithm,

$$\begin{aligned} \log \sqrt[5]{0.00327} &= \tfrac{1}{5} \log 0.00327 \\ &= \tfrac{1}{5}(\bar{3}.5145). \end{aligned}$$

Adding 5 to $\bar{3}.5145$, then indicating the subtraction,

$$\begin{aligned} \tfrac{1}{5}(\bar{3}.5145) &= \tfrac{1}{5}(2.5145 - 5) \\ &= 0.5029 - 1 \\ &= \bar{1}.5029. \end{aligned}$$

Now, $\bar{1}.5029 = \log 0.31855.$

Hence, $\sqrt[5]{0.00327} = 0.31855$, approximately.

If the logarithm with a negative characteristic is to be divided by some number, as in the above example, the number added to and subtracted from the logarithm should be exactly divisible by the divisor.

Thus, if the logarithm $\bar{3}.6217$ is to be divided by 7, the number added and subtracted may be 7, 14, or 70, or any other multiple of 7.

Similarly,

$$\begin{aligned} \tfrac{1}{4} \text{ of } \bar{3}.7248 &= \tfrac{1}{4}(37.7248 - 40) \\ &= 9.4312 - 10 \\ &= \bar{1}.4312. \end{aligned}$$

EXERCISES

Make the following computations by logarithms:

1. 71.6×0.327.
2. 3.86×24.3.
3. 1.068×0.0039.
4. 3.412×72.81.
5. 3.1416×8.16.
6. $0.144 \div 0.98$.
7. $681.7 \div 4.235$.
8. 1728×3.1416.
9. 1.414×0.0632.
10. 4617×0.00396.
11. 0.0786×0.0657.
12. $1.875 \div 0.1897$.
13. $498 \div 62.4$.
14. $9.12 \div 0.374$.
15. $0.019 \div 3.1416$.
16. $2.0037 \div 666$.
17. $7.29 \div 2.468$.
18. 0.996^6.
19. 1.009^{10}.
20. 1.786^5.
21. 0.06814^7.
22. 1.19^8.
23. $\sqrt[3]{4.68}$.
24. $\sqrt[3]{93.46}$.
25. $\sqrt[5]{9.2618}$.
26. $\sqrt[6]{0.09984}$.
27. $\sqrt[7]{20035}$.
28. $0.068^{\frac{5}{4}}$.
29. $(\frac{61}{27})^6$.
30. $(7\frac{5}{12})^4$.
31. $0.64 \times 7.23 \times 92.5$.
32. $82.6 \times 0.047 \times 1.289$.
33. $\sqrt{69 - \sqrt{1492}}$.
34. $\dfrac{2.476 \times 73.81}{0.524 \times 6184}$.
35. $\dfrac{5334 \times 0.0237}{47.4 \times 3.1416}$.
36. $\sqrt{\dfrac{628 \times 3.1416}{9.16^3}}$.
37. $\sqrt[3]{\dfrac{6.8 \times 0.785}{3.86^2}}$.
38. $\sqrt[5]{\dfrac{0.027^4 \times 32.6 \times \sqrt[4]{542}}{\sqrt[3]{412} \times \sqrt[5]{714} \times \sqrt[6]{628}}}$.

39. The circumference of a circle is $2\pi R$. Find the circumference of a circle in which $R = 36.5$ inches.

40. The area of a circle is πR^2. Find the area of a circle in which $R = 7.34$ inches.

41. The volume of a sphere is $\frac{4}{3}\pi R^3$. Find the volume of a sphere in which $R = 16.6$ inches.

42. Find the radius of a circle whose area is 92 square feet.

43. Find the radius of a sphere whose volume is 62 cubic feet.

44. In a right triangle the square of the hypotenuse equals the sum of the squares of the other two sides. Find the hypotenuse of a right triangle whose other two sides are 14 inches and 18 inches, respectively.

45. The area of a triangle $=\sqrt{s(s-a)(s-b)(s-c)}$, where a, b, and c are the sides, and $s=\frac{1}{2}(a+b+c)$.

Find the area of a triangle whose sides are 13 inches, 16 inches, and 21 inches respectively.

46. Find the area of a triangular building lot whose dimensions are 80 feet by 120 feet by 96 feet.

47. The formula $S_{2n}=\sqrt{2-\sqrt{4-S_n^2}}$ is used in computing the value of π.

If $S_n=1$, compute S_{2n} to four decimal places.

48. In the formula of Ex. 47, if $S_n=0.5176$, compute S_{2n} to four decimal places.

49. The distance of Venus from the sun is $93{,}000{,}000\sqrt[3]{\left(\frac{226}{365}\right)^2}$ miles. By simplifying this expression, find to the nearest hundred thousands of miles the distance of Venus from the sun.

50. In a steam engine, the average velocity v of the piston head, in inches per second, is computed by the formula $v=1.7\sqrt[3]{s\sqrt{\frac{1}{15}p}}$, where s is the distance over which the piston moves (in inches), and p the number of pounds of pressure of steam in the cylinder.

If $s=28.75$ inches and $p=120$ pounds, find v.

51. The area A of the cross section of a chimney, in square feet, required to carry off the smoke is computed by the formula

$$A=\frac{0.06\,P}{\sqrt{h}},$$

where P is the number of pounds of coal burned per hour, and h is the height of the chimney in feet.

What should be the area of the cross section of a chimney 72 feet high to carry off the smoke if 750 pounds of coal are burned per hour?

52. If p dollars are invested at $r\%$ compound interest, compounded annually, the value, or *amount*, at the end of t years will be $p(1+r)^t$.

Find the amount at the end of 6 years of \$6000 compounded annually at 4%.

53. Find the amount at the end of 8 years of \$5500 compounded annually at $3\frac{1}{2}\%$.

54. If the interest is compounded semiannually, the formula for the amount in Ex. 52 becomes $p(1+\frac{1}{2}r)^{2t}$.

I invest \$200 in a savings bank that pays 3% interest compounded semiannually. To what will the sum amount in 10 years?

156. Exponential Equations. — An equation in which the unknown number occurs as an exponent is called an **exponential equation.**

The root of an exponential equation may be obtained by the aid of logarithms.

EXAMPLE 1. — Solve $5^x = 267$.

Taking the logarithms of both members,

$$x \log 5 = \log 267.$$

Hence,
$$x = \frac{\log 267}{\log 5} = \frac{2.4265}{0.6990} = 3.47^+.$$

EXAMPLE 2. — Solve $5^{3x-1} = 4^{x-1}$.

Taking the logarithms of both members,

$$(3x-1)\log 5 = (x-1)\log 4.$$

Removing parentheses, transposing, etc.,

$$\begin{aligned} x &= \frac{\log 5 - \log 4}{3\log 5 - \log 4} \\ &= \frac{0.6990 - 0.6021}{2.0970 - 0.6021} \\ &= 0.0648. \end{aligned}$$

EXERCISES

Solve:

1. $3^x = 81$.
2. $5^x - 25 = 0$.
3. $2^{2x} = 64$.
4. $3^x = 15$.
5. $2^{3x} = 25$.
6. $4^{x-1} = 3^{x+1}$.
7. $2^{x-1} = 0.3^{2x-5}$.
8. $5^{2x} - 12^{x+1} = 0$.
9. $4^{x+1} = 8 \cdot 2^{x+2}$.
10. $\sqrt[3]{2^{x+2}} = \sqrt{2^{x-3}}$.
11. $\sqrt{3^{11-x}} = 3^{8-x}$.
12. $\sqrt[3]{3^{x+7}} = 2^x$.

13. In how many years will $1 double itself at 4 %, interest compounded annually?

SUGGESTION. — See formula in Ex. 52, § 155. If t is the number of years required, show that $1.04^t = 2$.

14. In how many years will $1 double itself at 3 %, interest compounded annually?

15. In how many years will $1 double itself at 4 %, interest compounded semiannually? (See formula in Ex. 54, § 155.)

16. In how many years will $ 2000 amount to $ 5309.41, interest compounded annually at 5 %?

SUPPLEMENTARY EXERCISES

1. Show that the logarithm of 1 to any base is 0.
2. Show that the logarithm of the base itself is 1.
3. The logarithm of $\frac{1}{x}$ is called the **cologarithm** of x. Show that $\text{colog}\, x = - \log x$.

Find the values of:

4. $$\sqrt[3]{\frac{27^4 \times 0.028^2 \times 16.75^3}{\sqrt{629} \times \sqrt[4]{8710} \times \sqrt[5]{126.3}}}.$$

5. $$98.7\sqrt[5]{\frac{0.068 \times \sqrt[3]{28.59}}{\sqrt[3]{6} \times \sqrt[6]{206.4} \times \sqrt{0.0091}}}.$$

Solve:

6. $\sqrt[x]{6.82} = 2.36$.
7. $3^{2x} - 4(3^x) - 12 = 0$.
8. $2^{4x} - 3(2^{2x}) = 4$.
9. $2(4^{2x}) - 4^x - 6 = 0$.

10. The sum of money p that placed at interest compounded annually will amount to a given sum a in a number of years at a given rate is called the *present worth* of a. If p dollars at r % will amount to a dollar in t years, interest compounded annually, then $p = \frac{a}{(1+r)^t}$.

Find the present worth of $12,000 at 3 %, interest compounded annually for 10 years.

11. Find the present worth of $2400 at 4 %, interest compounded annually for 6 years.

12. Find the present worth of $3200 at 3½ %, interest compounded annually for 12 years.

MISCELLANEOUS EXERCISES

Factor:

1. $(9n - a^2)^2 - a^2$.
2. $(2a - b)^2 - (a + 2b)^2$.
3. $4x^2 - 4a^2 + 4at - t^2$.
4. $N^4 + 9N^2 + 81$.
5. $V^5 - 11V^3 + 30V$.
6. $4k^2 + 6k + 2$.
7. $30R^2 + 56Rr + 24r^2$.
8. $p^4 - 11p^2q^2 + q^4$.
9. $12N^2 - 5N - 2$.
10. $2v^4 - 3v^2 - 14$.
11. $x^8 - 1$.
12. $a^5b - ab^5$.
13. $V^{16} - T^{16}$.
14. $m^{12} + n^{12}$.
15. $1 + 8a^9$.
16. $a^2 - ab - b - 1$.
17. $t^3 - t^2 - t + 1$.
18. $4 + 9y^4 - 37y^2$.
19. $(n^2 - n)^3 - 8$.
20. $H^3 + H^2 - H - 1$.
21. $v^3 - 7v^2 + 14v - 8$.
22. $a^2b^2 - a^2 - b^2 + 1$.
23. $(a^2 + b^2 - c^2)^2 - 4a^2b^2$.
24. $v^4 - v^2 + 12vw - 36w^2$.
25. $R^3 + R + N^3 + N$.
26. $h^2 - t^2 - h - t$.
27. $n^3 - 6n^2 + 11n - 6$.
28. $A^3 - 27A - 10$.
29. $m^5 - 32n^{10}$.
30. $x^5 - x^4 - x^3 + x$.
31. $e^{3n} + e^{2n} + e^n + 1$.
32. $3G^2 + (3a + b)G + ab$.

33. $ab^2 + ac^2 + a^2b + a^2c + bc^2 + b^2c + 2\,abc.$

34. $25\,n^2 - 20\,n + 4 - 4\,a^2 - 9\,b^2 - 12\,ab.$

35. $6\,x^{2n+6} - 25\,x^{n+3}y^{n-1} + 4\,y^{2n-2}.$

36. $xy^3 - x^3y + xz^3 - x^3z + yz^3 - y^3z.$

37. $n^2 + (a + b + c)\,n + ab + ac.$

38. $x^2 + a^2 - (b^2 + c^2) - 2(bc - ax).$

39. $mt^3 + ant^2 - b^2nt - ab^2m + nt^3 + amt^2 - b^2mt - ab^2n.$

40. Subtract $3\,n^3 - 4\,n + 2$ from $5\,n^2 - 2\,n - 1$, then add the difference to $5\,n^2 + 4$.

41. Simplify $a - [-3\,a - (a - 5) - (4 - 2\,a)]$.

42. Divide $a^2 + ab - 3\,ac - 2\,b^2 + 9\,bc - 10\,c^2$ by $a - b + 2\,c$.

43. Simplify $(r^n + 1 - r^{\frac{1}{n}})(r^{2n} - r^n - r^{\frac{1}{n}} - r^{\frac{2}{n}})$.

44. Find the H. C. F. of $3\,t^3 + 7\,t^2 + 3\,t + 2$ and $t^3 + 5\,t^2 + 6\,t$.

45. Find the L. C. M. of $7\,mn^2(x^3 - 3\,m^2x + 2\,m^3)$, $25\,n^3(x^2 - m^2)^2$ and $35\,m^3n\,(x^2 + mx - 2\,m^2)$.

46. Find the H. C. F. and the L. C. M. of $x^4 - y^4$, $x^3 + y^3$, and $x^3 + 2\,x^2y + 2\,xy^2 + y^3$.

47. Is $a^{15} + b^{15}$ divisible by $a + b$? By $a - b$? Give proof.

48. Find the L. C. M. of $2p^3 - 5p^2 - 22p - 15$ and $6p^4 - 21p^3 - 41p^2 - 14p - 30$.

49. Find the value of n for which $a^2 - a + 1$ is a factor of $6\,a^4 - 2\,a^3 + 2\,na^2 + 2\,a + n$.

Find the square root of:

50. $9 - 6\,n - 29\,n^2 + 22\,n^3 + 21\,n^4 - 20\,n^5 + 4\,n^6.$

51. $1 + 4\,y^3 - 2\,y^2 - 4\,y^5 - y^4 + 6\,y^6 - 4\,y^7 + y^8.$

52. $x^{-4} - 2\,x^{-3}y^{\frac{1}{2}} - 2\,x^{\ 1}y^{\frac{3}{2}} + 3\,x^{-2}y + y^2.$

53. What fraction must be added to $\dfrac{4\,n}{(n + 1)(n - 1)^2} - \dfrac{n + 1}{(n - 1)^2}$ to make it equal to 1?

Simplify:

54. $\dfrac{k+1}{12k^3+8k^2-3k-2} \div \dfrac{2k+1}{12k^3-10k^2-6k+4}$.

55. $\dfrac{1}{r^3-r} - \left(\dfrac{1}{r^3-1} - \dfrac{1}{r^3+1}\right)$.

56. $\dfrac{x^2(y-z)^3+y^2(z-x)^3+z^2(x-y)^3}{xy+yz+zx}$.

57. $\left(\dfrac{m-n}{m+n} - \dfrac{m^2-n^2}{m^2+n^2}\right) \div \left(\dfrac{m+n}{m-n} + \dfrac{m^2+n^2}{m^2-n^2}\right)$.

58. $\dfrac{x^2-1}{y+y^2} \times \left(1+\dfrac{y}{1-y}\right) \times \dfrac{y^2-1}{x+1}$.

59. $\dfrac{1-\left(\dfrac{1}{1-n} - \dfrac{m^2+n^2-m+n}{1-n^2}\right)}{\left(\dfrac{m}{1+m}-1\right)\left(\dfrac{1-m^2}{1+n}\right)}$.

60. If $\dfrac{a}{b+c}=x$, $\dfrac{b}{a+c}=y$, $\dfrac{c}{a+b}=z$, find the value of $\dfrac{x}{x+1}+\dfrac{y}{y+1}+\dfrac{z}{z+1}$.

61. If $a=\dfrac{m+n}{m-n}$ and $b=\dfrac{m-n}{m+n}$, find the value of $\dfrac{a^2+b^2}{a^2-b^2}$.

62. Simplify $\dfrac{\dfrac{a+b}{ab}\left(\dfrac{1}{a}-\dfrac{1}{b}\right)-\dfrac{b+c}{bc}\left(\dfrac{1}{c}-\dfrac{1}{b}\right)}{\dfrac{a+c}{ac}\left(\dfrac{1}{a}-\dfrac{1}{c}\right)}$.

63. Show that the equation $\dfrac{2nt}{t+2n^2+6n}=\dfrac{1}{\dfrac{t}{n^2+t^2-9}+\dfrac{n+3}{t}}$ reduces to $t^2(t^2-2nt+n^2-9)=0$.

64. If $5y^2-3y+(3-n)$ is a perfect square, what is the value of n?

65. For what value of n is $R^2-(6n-1)R+9n^2$ a perfect square?

66. Solve $x^2 - 1.4\,x + 0.2 = 0$, and compute the larger root to three decimal places.

67. Find to three decimal places the value of the larger root of $2\,x^2 - 0.5 \times 1.2 = 0$.

68. Form the equation whose two roots are $3 + \sqrt{5}$ and $3 - \sqrt{5}$.

69. Find, without solving, the nature of the roots of $65\,x^2 - 19\,x + 2 = 0$.

Solve:

70. $x^2 + 8\,x + 6\sqrt{x^2 + 8\,x - 8} = 3$.

71. $\sqrt{2\,n + 6} - \sqrt{n - 4} = \sqrt{n + 4}$.

72. $\sqrt{3\,r + 4} + \sqrt{3\,r - 5} - 9 = 0$.

73. Solve the system $\begin{cases} 2\,v^2 + vt = 18, \\ t^2 - 2\,vt = 5. \end{cases}$

74. Solve the system $\begin{cases} x^2 + xy + y^2 = 49, \\ x^2 - xy + y^2 = 19. \end{cases}$

75. Solve the system $\begin{cases} a + b + 2\,c = 1, \\ 2\,a + 3\,b + 2\,c = 4, \\ 4\,a + 9\,b + 2\,c = 16. \end{cases}$

76. In the formula $S = \frac{1}{2}\,n\,[2\,a + (n - 1)\,d]$, solve for n in terms of the other letters.

Rationalize the divisors of:

77. $\dfrac{1}{1 + \sqrt{2}}$.   **78.** $\dfrac{\sqrt{5} - \sqrt{3}}{\sqrt{5} + \sqrt{3}}$.   **79.** $\dfrac{1}{a + \sqrt{ab} + b}$.

80. Find the value to two places of decimals of $\dfrac{9 - 2\sqrt{5}}{4 + \sqrt{5}}$.

81. Divide $\sqrt[3]{6}$ by $\sqrt{2}$, and express the result in its simplest form.

Simplify:

82. $(ab^{-2}c^3)^{\frac{1}{2}} \times (a^6b^3c^{-2})^{\frac{1}{3}}$.

83. $(256\,m^{\frac{2}{3}}n^{-\frac{4}{3}})^{-\frac{3}{4}}$.

84. $\dfrac{a^{\frac{1}{2}}b^{-\frac{3}{2}}}{a^{\frac{2}{3}}b^{-\frac{1}{4}}}$.

85. $\left(\dfrac{x^{-2}y^{-4}z^{-1}}{x^3y^2z^{-2}}\right)^{-3}$.

86. $\sqrt{\dfrac{a}{b^3}} \times \sqrt[3]{\dfrac{b^2}{c^2}} \times \sqrt{\dfrac{ab}{c}}$.

87. Solve $\dfrac{\sqrt{2+x}}{\sqrt{2}+\sqrt{2+x}} = \dfrac{\sqrt{2-x}}{\sqrt{2}-\sqrt{2-x}}$.

88. If $\dfrac{a}{b} = \dfrac{c}{d}$, show that $\dfrac{a^2+b^2}{a(a-b)} = \dfrac{c^2+d^2}{c(c-d)}$.

89. If $\dfrac{a}{b} = \dfrac{c}{d}$, show that $ab+cd$ is a mean proportional between a^2+c^2 and b^2+d^2.

90. Show that either root of $x^2 - q = 0$ is a mean proportional between the roots of $x^2 + px + q = 0$.

91. If m is a mean proportional between a and b, show that $\dfrac{a-b}{2a-m-b} = \dfrac{a+m}{2a+m}$.

92. The second term of an A. P. is 10 and the seventh term is 25. Find the fiftieth term.

93. Insert three arithmetical means between 6 and 10.

94. Insert n arithmetical means between a and b.

95. Find the sum of 40 terms of the series 12, 9, 6, ….

96. In the series 15, $12\frac{1}{2}$, 10, …, which term is $-27\frac{1}{2}$?

97. How many terms of the series $3\frac{1}{2}$, 5, $6\frac{1}{2}$, … must be added to give 498?

98. Find the eighth term of 2, 3, $4\frac{1}{2}$, ….

99. Insert three geometrical means between 6 and 486.

100. Insert three geometrical means between a and b.

101. Find the sum of 6 terms of $3\sqrt{2}$, 6, $6\sqrt{2}$, ….

102. At each stroke of an air pump 10 % of the air in the bell jar is removed, and the air remaining expands and completely fills the jar again. The capacity of the jar is 360 cu. in. What part of the original quantity of air remains in the jar at the end of the eighth stroke?

103. Find the common fraction to which 6.4545 ⋯ is equal.

104. A flywheel whose circumference is 8 ft. makes 120 revolutions per minute. If, when the power is shut off, the wheel makes 98 % as many revolutions each second thereafter as it did the preceding second, how far will a belt on the wheel run by the time it is about to stop?

105. Find the common ratio of an infinite G. P. of which the first term is 1 and the sum of the terms $1\frac{1}{2}$.

106. Find the twelfth term in the expansion of $\left(\frac{\sqrt{a}}{2}-\frac{2\sqrt{b}}{3}\right)^{22}$.

107. Find by the Binomial Theorem the cube root of 121 to three places of decimals.

108. Expand $(1-n^{-\frac{1}{2}})^{-\frac{1}{2}}$ to five terms.

109. Simplify $\frac{(a^2bcd^{-1})^l(b^2acd^{-2})^m(c^2abd^{-3})^n}{(a^{-2}bcd)^l(b^{-2}acd^2)^m(c^{-2}abd^3)^n}\times\frac{(d^2)^{l+2m+3n}}{(a^lb^mc^n)^2}$.

110. Give the numerical values of $\sqrt[3]{-8}$, $27^{-\frac{1}{3}}$, $(\sqrt{-1})^6$.

111. Simplify $\sqrt[m]{a^{-x}}\div\sqrt[n]{a^{-y}}$.

112. If $a^{-\frac{7}{8}}=b$, $b^{-1\frac{3}{4}}=c$, and $c^{-\frac{8}{3}}=a^{\frac{1}{2}}$, find a.

113. Simplify $\frac{8m-n}{2m^{\frac{1}{3}}-n^{\frac{1}{3}}}-2m^{\frac{1}{3}}n^{\frac{1}{3}}$.

114. In $a^2-2ab+b^2-4\sqrt{2}(a+b)+8$, substitute for a and b the values $a=\frac{m+n+1}{\sqrt{2}}$, $b=\frac{m-n+1}{\sqrt{2}}$, and simplify.

115. Show that if $n=\sqrt{a^2-b^2}$,

$$\log n=\tfrac{1}{2}\log(a+b)+\tfrac{1}{2}\log(a-b).$$

116. If $\log 3 = 0.4771$ and $\log 5 = 0.6990$, solve $3^{x+7} = 5^x$.

117. If $\log 2 = 0.3010$ and $\log 3 = 0.4771$, find the value of $\log \sqrt[5]{0.36}$.

118. The velocity of sound in a gas is computed by the formula $V = \sqrt{\frac{E}{D}} \times 1.41$. Solve for E. For D.

119. Eliminate R from the formulas $C = \frac{E}{R}$ and $C' = \frac{E}{R + r}$.

120. If $\frac{g}{g'} = \frac{\frac{1}{R^2}}{\frac{1}{(R+h)^2}}$, solve for h.

121. Solve $t = \frac{u + \sqrt{u^2 - 2\,gs}}{g}$ for u.

122. The velocity v at which water is discharged from a pipe is found from the formula

$$4\,v^2 + 5\,v - 2 = \frac{1200\,Hd}{l}.$$

If $H = 60$, $d = 6$, and $l = 1600$, find v.

123. If $y = 2\,\pi r(r - \sqrt{r^2 - R^2})$, solve for r. Find the value of y when $r = \frac{1}{2}$ and $R = 0.1$.

124. If $V = \frac{1}{6}\,h\{\frac{1}{2}\,b(e' + e'') + 4\,M\}$ and $M = \frac{1}{4}\,b(\frac{1}{2}[e + e'] + \frac{1}{2}[e + e''])$, show that $V = \frac{1}{6}\,hb(e + e' + e'')$.

125. If $\frac{S}{\frac{1}{2}(S+s)} = \frac{\sqrt{B}}{\sqrt{M}}$ and $\frac{s}{\frac{1}{2}(S+s)} = \frac{\sqrt{b}}{\sqrt{M}}$, show that

$$4\,M = B + b + 2\sqrt{Bb}.$$

126. By rationalizing the denominators and simplifying, show that $F = \frac{1}{3}\,B \times \frac{H\sqrt{B}}{\sqrt{B} - \sqrt{b}} - \frac{1}{3}\,b \times \frac{H\sqrt{B}}{\sqrt{B} - \sqrt{b}}$ reduces to

$$F = \frac{1}{3}\,H(B + b + \sqrt{Bb}).$$

N.	0	1	2	3	4	5	6	7	8	9
10	0000	0043	0086	0128	0170	0212	0253	0294	0334	0374
11	0414	0453	0492	0531	0569	0607	0645	0682	0719	0755
12	0792	0828	0864	0899	0934	0969	1004	1038	1072	1106
13	1139	1173	1206	1239	1271	1303	1335	1367	1399	1430
14	1461	1492	1523	1553	1584	1614	1644	1673	1703	1732
15	1761	1790	1818	1847	1875	1903	1931	1959	1987	2014
16	2041	2068	2095	2122	2148	2175	2201	2227	2253	2279
17	2304	2330	2355	2380	2405	2430	2455	2480	2504	2529
18	2553	2577	2601	2625	2648	2672	2695	2718	2742	2765
19	2788	2810	2833	2856	2878	2900	2923	2945	2967	2989
20	3010	3032	3054	3075	3096	3118	3139	3160	3181	3201
21	3222	3243	3263	3284	3304	3324	3345	3365	3385	3404
22	3424	3444	3464	3483	3502	3522	3541	3560	3579	3598
23	3617	3636	3655	3674	3692	3711	3729	3747	3766	3784
24	3802	3820	3838	3856	3874	3892	3909	3927	3945	3962
25	3979	3997	4014	4031	4048	4065	4082	4099	4116	4133
26	4150	4166	4183	4200	4216	4232	4249	4265	4281	4298
27	4314	4330	4346	4362	4378	4393	4409	4425	4440	4456
28	4472	4487	4502	4518	4533	4548	4564	4579	4594	4609
29	4624	4639	4654	4669	4683	4698	4713	4728	4742	4757
30	4771	4786	4800	4814	4829	4843	4857	4871	4886	4900
31	4914	4928	4942	4955	4969	4983	4997	5011	5024	5038
32	5051	5065	5079	5092	5105	5119	5132	5145	5159	5172
33	5185	5198	5211	5224	5237	5250	5263	5276	5289	5302
34	5315	5328	5340	5353	5366	5378	5391	5403	5416	5428
35	5441	5453	5465	5478	5490	5502	5514	5527	5539	5551
36	5563	5575	5587	5599	5611	5623	5635	5647	5658	5670
37	5682	5694	5705	5717	5729	5740	5752	5763	5775	5786
38	5798	5809	5821	5832	5843	5855	5866	5877	5888	5899
39	5911	5922	5933	5944	5955	5966	5977	5988	5999	6010
40	6021	6031	6042	6053	6064	6075	6085	6096	6107	6117
41	6128	6138	6149	6160	6170	6180	6191	6201	6212	6222
42	6232	6243	6253	6263	6274	6284	6294	6304	6314	6325
43	6335	6345	6355	6365	6375	6385	6395	6405	6415	6425
44	6435	6444	6454	6464	6474	6484	6493	6503	6513	6522
45	6532	6542	6551	6561	6571	6580	6590	6599	6609	6618
46	6628	6637	6646	6656	6665	6675	6684	6693	6702	6712
47	6721	6730	6739	6749	6758	6767	6776	6785	6794	6803
48	6812	6821	6830	6839	6848	6857	6866	6875	6884	6893
49	6902	6911	6920	6928	6937	6946	6955	6964	6972	6981
50	6990	6998	7007	7016	7024	7033	7042	7050	7059	7067
51	7076	7084	7093	7101	7110	7118	7126	7135	7143	7152
52	7160	7168	7177	7185	7193	7202	7210	7218	7226	7235
53	7243	7251	7259	7267	7275	7284	7292	7300	7308	7316
54	7324	7332	7340	7348	7356	7364	7372	7380	7388	7396

N.	0	1	2	3	4	5	6	7	8	9
55	7404	7412	7419	7427	7435	7443	7451	7459	7466	7474
56	7482	7490	7497	7505	7513	7520	7528	7536	7543	7551
57	7559	7566	7574	7582	7589	7597	7604	7612	7619	7627
58	7634	7642	7649	7657	7664	7672	7679	7686	7694	7701
59	7709	7716	7723	7731	7738	7745	7752	7760	7767	7774
60	7782	7789	7796	7803	7810	7818	7825	7832	7839	7846
61	7853	7860	7868	7875	7882	7889	7896	7903	7910	7917
62	7924	7931	7938	7945	7952	7959	7966	7973	7980	7987
63	7993	8000	8007	8014	8021	8028	8035	8041	8048	8055
64	8062	8069	8075	8082	8089	8096	8102	8109	8116	8122
65	8129	8136	8142	8149	8156	8162	8169	8176	8182	8189
66	8195	8202	8209	8215	8222	8228	8235	8241	8248	8254
67	8261	8267	8274	8280	8287	8293	8299	8306	8312	8319
68	8325	8331	8338	8344	8351	8357	8363	8370	8376	8382
69	8388	8395	8401	8407	8414	8420	8426	8432	8439	8445
70	8451	8457	8463	8470	8476	8482	8488	8494	8500	8506
71	8513	8519	8525	8531	8537	8543	8549	8555	8561	8567
72	8573	8579	8585	8591	8597	8603	8609	8615	8621	8627
73	8633	8639	8645	8651	8657	8663	8669	8675	8681	8686
74	8692	8698	8704	8710	8716	8722	8727	8733	8739	8745
75	8751	8756	8762	8768	8774	8779	8785	8791	8797	8802
76	8808	8814	8820	8825	8831	8837	8842	8848	8854	8859
77	8865	8871	8876	8882	8887	8893	8899	8904	8910	8915
78	8921	8927	8932	8938	8943	8949	8954	8960	8965	8971
79	8976	8982	8987	8993	8998	9004	9009	9015	9020	9025
80	9031	9036	9042	9047	9053	9058	9063	9069	9074	9079
81	9085	9090	9096	9101	9106	9112	9117	9122	9128	9133
82	9138	9143	9149	9154	9159	9165	9170	9175	9180	9186
83	9191	9196	9201	9206	9212	9217	9222	9227	9232	9238
84	9243	9248	9253	9258	9263	9269	9274	9279	9284	9289
85	9294	9299	9304	9309	9315	9320	9325	9330	9335	9340
86	9345	9350	9355	9360	9365	9370	9375	9380	9385	9390
87	9395	9400	9405	9410	9415	9420	9425	9430	9435	9440
88	9445	9450	9455	9460	9465	9469	9474	9479	9484	9489
89	9494	9499	9504	9509	9513	9518	9523	9528	9533	9538
90	9542	9547	9552	9557	9562	9566	9571	9576	9581	9586
91	9590	9595	9600	9605	9609	9614	9619	9624	9628	9633
92	9638	9643	9647	9652	9657	9661	9666	9671	9675	9680
93	9685	9689	9694	9699	9703	9708	9713	9717	9722	9727
94	9731	9736	9741	9745	9750	9754	9759	9763	9768	9773
95	9777	9782	9786	9791	9795	9800	9805	9809	9814	9818
96	9823	9827	9832	9836	9841	9845	9850	9854	9859	9863
97	9868	9872	9877	9881	9886	9890	9894	9899	9903	9908
98	9912	9917	9921	9926	9930	9934	9939	9943	9948	9952
99	9956	9961	9965	9969	9974	9978	9983	9987	9991	9996

TABLE OF SQUARE ROOTS AND CUBE ROOTS

Nos.	Square Roots	Cube Roots	Nos.	Square Roots	Cube Roots
1	1.000	1.000	51	7.141	3.708
2	1.414	1.259	52	7.211	3.732
3	1.732	1.442	53	7.280	3.756
4	2.000	1.587	54	7.348	3.779
5	2.236	1.709	55	7.416	3.802
6	2.449	1.817	56	7.483	3.825
7	2.645	1.912	57	7.549	3.848
8	2.828	2.000	58	7.615	3.870
9	3.000	2.080	59	7.681	3.892
10	3.162	2.154	60	7.745	3.914
11	3.316	2 223	61	7.810	3.936
12	3.464	2.289	62	7.874	3.957
13	3.605	2.351	63	7.937	3.979
14	3.741	2.410	64	8.000	4.000
15	3.872	2.466	65	8.062	4.020
16	4.000	2.519	66	8.124	4.041
17	4.123	2.571	67	8.185	4.061
18	4.242	2.620	68	8.246	4.081
19	4.358	2.668	69	8.306	4.101
20	4.472	2.714	70	8.366	4.121
21	4.582	2.758	71	8.426	4.140
22	4.690	2.802	72	8.485	4.160
23	4.795	2.843	73	8.544	4.179
24	4.898	2.884	74	8.602	4.198
25	5.000	2.924	75	8.660	4.217
26	5.099	2.962	76	8.717	4.235
27	5.196	3.000	77	8.774	4.254
28	5.291	3.036	78	8.831	4.272
29	5.385	3.072	79	8.888	4.290
30	5.477	3.107	80	8.944	4.308
31	5.567	3.141	81	9.000	4.326
32	5.656	3.174	82	9.055	4.344
33	5.744	3.207	83	9.110	4.362
34	5.830	3.239	84	9.165	4.379
35	5.916	3.271	85	9.219	4.396
36	6.000	3.301	86	9.273	4.414
37	6.082	3.332	87	9.327	4.431
38	6.164	3.361	88	9.380	4.447
39	6.244	3.391	89	9.433	4.464
40	6.324	3.419	90	9.486	4.481
41	6.403	3.448	91	9.539	4.497
42	6.480	3.476	92	9.591	4.514
43	6.557	3.503	93	9.643	4.530
44	6.633	3.530	94	9.695	4.546
45	6.708	3.556	95	9.746	4.562
46	6.782	3.583	96	9.797	4.578
47	6.855	3.608	97	9.848	4.594
48	6.928	3.634	98	9.899	4.610
49	7.000	3.659	99	9.949	4.626
50	7.071	3.684	100	10.000	4.641

INDEX

(Numbers refer to Sections)

Addition, 3, 4, 5, 6, 72, 80.
  elimination by, 54.
Antecedent, 112.
Arithmetical means, 130.
  progressions, 128.
Axioms, 43.
Axis, 51.

Binomial, 2.
  Theorem, 142.

Characteristic, 149.
Common denominator, 34.
  factor, highest, 28.
  multiple, lowest, 29.
Comparison, elimination by, 54.
Completing the square, 89.
Complex expression, 78.
  fraction, 39.
Conditional equation, 40.
Consequent, 112.
Constant, 115.
Continued fraction, 39.
Coördinates, 51.
Cube root, 62.
Cubic equation, 42.

Defective system, 107.
Degree of equation, 42.
Denominator, 30.
  common, 34.
Determinant, 58.
Determinate system, 56.
Direct variation, 116.
Discriminant, 93.
Dissimilar terms, 5.
Distribution, law of, 11.
Division, 16, 17, 18, 75, 83.
  synthetic, 18.

Elimination, 54.
  by addition or subtraction, 54.
  by comparison, 54.
  by substitution, 54.
Ellipse, 109.
Equations, 40.
  conditional, 40.
  cubic, 42.
  degree of, 42.
  discriminant, 93.
  equivalent, 44, 56.
  exponential, 156.
  fractional, 41.
  graph of, 51.
  homogeneous, 104.
  identical, 40.
  impossible, 100.
  inconsistent, 56.
  integral, 41.
  linear, 42.
  literal, 48.
  quadratic, 42.
  symmetrical, 106.
  system of, 52.
Equivalent equations, 44, 56.
Exponential equations, 156.
Exponents, fractional, 140.
  laws of, 13, 16, 137, 141.
  negative, 139.
  positive integral, 137.
  zero, 138.
Expressions, complex, 78.
  fractional, 36.
  integral, 36.
  literal, 2.
  mixed, 36.
  number, 2.
  radical, 69.
Extraneous roots, 100.
Extremes, 113.

Factoring, 19.
  general directions for, 27.
Factors, 19.
  highest common, 28.
  prime, 19.
  rationalizing, 75.
Formulas, 48.

Fourth proportional, 113.
Fractional equations, 41.
  exponents, 140.
  expressions, 36.
Fractions, 30.
  complex, 39.
  continued, 39.
  denominator of, 30.
  indeterminate, 125.
  numerator of, 30.
  reduction of, 33, 34.
  signs of, 31.
  terms of, 30.

Geometrical means, 134.
  progressions, 132.
Graph of an equation, 51.
Grouping, law of in addition, 3.
  law of in multiplication, 11.
  signs of, 10.

Highest common factor, 28.
Homogeneous equations, 104.
Hyperbola, 109.

Identical equations, 40.
Identity, 40.
Imaginary numbers, 63.
  typical form of, 79.
  unit, 79.
Impossible equations, 100.
  systems of, 56.
Inconsistent equations, 56.
Indeterminate fractions, 125.
  systems, 56.
Index of a root, 62.
Infinity, 121.
Integral equations, 41.
  expressions, 36.
Inverse variation, 117.
Isolating a surd, 99.

Law of distribution, 11.
  of grouping in addition, 3.
  of grouping in multiplication, 11.
  of order in addition, 3.
  of order in multiplication, 11.
Like terms, 5.
Limit of a variable, 120.
Linear equations, 42.
Literal equations, 48.
Literal expressions, 2.
  numbers, 2.
Logarithms, 147.
  characteristic of, 149.
  common, 148.
  mantissa of, 149.
Lowest common multiple, 29.

Mantissa, 149.
Mean proportional, 113.
Means, 113.
  arithmetical, 130.
  geometrical, 134.
Mixed expressions, 36.
Monomial, 2.
Multiple, lowest common, 29.
Multiplication, 11, 12, 13, 14, 15, 74 82

Negative exponent, 139.
Number expression, 2.
  imaginary, 63.
  literal, 2.
Numerator, 30.

Order, law of in addition, 3.
  law of in multiplication, 11.
  of a determinant, 58, 59.
  of a surd, 69.

Parabola, 109.
Pascal's Triangle, 142.
Polynomial, 2.
Principal root, 64.
Products, special, 15.
Progressions, 127.
  arithmetical, 128.
  geometrical, 132.
Proportion, 113.
  terms of, 113.
Proportional, fourth, 113.
  mean, 113.
  third, 113.

Quadratic equations, 42.
  complete, 86.
  pure, 86.
Quadratic surd, 69.

Radical expressions, 69.
  sign, 62.
Ratio, 112.

Rationalizing factor, 75.
Real numbers, 63.
Reduction of fractions, 33, 34.
Remainder Theorem, 25.
Root, extraneous, 100.
  index of, 62.
  principal, 64.
  square, cube, etc., 62.

Series, 126.
Signs of grouping, 10.
Similar terms, 5.
Simplest form of surd, 70.
Special products, 15.
Square, completing the, 89.
  root, 62.
Substitution, elimination by, 54.
Subtraction, 7, 8, 9, 72, 80.
  elimination by, 54.
Surd, 69.
  isolating a, 99.
  order of, 69.
  quadratic, 69.
  simplest form of, 70.
Symmetrical equations, 106.
Synthetic division, 18.
System of equations, 52.
  defective, 107.
System of equations, determinate, 56.
  impossible, 56.
  indeterminate, 56.
  solution of, 52.

Terms, 2.
  dissimilar, 5.
  of a fraction, 30.
  of a proportion, 113.
  of a series, 126.
  similar, 5.
Theorem, Binomial, 142.
  Remainder, 25.
Third proportional, 113.
Transposition, 45.
Triangle, Pascal's, 142.
Trinomial, 2.
Typical form of imaginary number, 79.

Unit, imaginary, 79.

Variable, 115.
  limit of, 120.
Variation, direct, 116.
  inverse, 117.
  joint, 118.

Zero exponent, 138.

ANSWERS

Pages 2, 3.

1. By writing them side by side, with the sign ×, a dot, or no sign at all between them.

2. $5n$; xy; $2ab$.

3. By enclosing each in parentheses and writing these side by side, as in Ex. 1.

4. $(t-4)(2t+3)$.

5. aa; nnn; $(x-y)(x-y)(x-y)(x-y)$.

6. Exponent; base; a power.

7. Numerical coefficient.

8. 2; 12; 6.

9. No; $3a=12$, $a^3=64$; $3a=30$, $a^3=1000$.

10. 1, monomial; 2, binomial; 3, trinomial; 2, binomial; 4, polynomial; 1, monomial; 1, monomial; 2, binomial; 4, polynomial; 5, polynomial.

11. 81; 256.
12. 60.
13. 16.
14. 49.
15. 179.
16. 528.
17. 64.
18. 113.0976 cu. in.
19. 132 sq. in.
20. 7439.3088 cu. in.

Page 4.

1. 20.
2. 20.
3. 19.
4. 20.
5. 30.
6. 36.
7. 2300.
8. 2487.
9. 2092.
10. 25,004.

Page 5.

1. -8; $+15$; -6; $+2$; -2; -18.

2. -8.
3. $+5$.
4. -5.
5. -2.
6. $+6$.
7. $+3$.

Pages 5, 6.

1. $+3a$.
2. $-7n$.
3. $-15W$.
4. $+12t^2$.
5. $+2a^2b^3$.
6. $-6N$.
7. $+9x^3$.
8. $-4ab$.
9. $+4Vt$.
10. $-7y^4$.
11. $-4k$.
13. $3a^2+2ab-2b^2$.
14. $6x-3$.
15. $-3A^3-5B^3+3A^2B$.
16. $-3y^3-5y^2-4y-10$.
17. $7a-5b-c$.
18. $3w^2+2wv-4v^2$.
19. 160.
20. 520.
21. 1440.
22. 4800.
23. 1440.
24. 6500.
25. 1860.
26. 314.16.
27. 62.832.

Page 7.

1. $2a-8b-4c$.
2. $3a^2+b^2$.
3. $3x^2-5x+4$.
4. $2a^3+4a^2b-2ab^2$.
5. $-3a^2+10a+3$.
6. $28+2t+t^2$.
7. $3x^2+2xy+3y^2$.
8. $a^3+3a^2+2-5ab+b^2$.
9. $21+12v+2v^2$.
10. $2a$.
11. $4x^2+2x+5$.
12. $-3a^2+2a$.
13. $2x^2-2xy+5y^2$.

Pages 7, 8.

1. -2; $+4$; $+7$; -6; $+4$; $+7$.
2. -27; $+12$; -6; -15; -3; -4.
3. $+6$; $+36$; $+6$; $+40$; $+14$; $+7$.
4. $+2$; -9; -6.
5. $+18$; $+7$; $+4$; $+19$; $+14$; $+24$.
6. $-6°$.
7. 110.
8. 63.
9. 56.
10. -5 subtracted from -6; -1.
11. 12; 6; -10.

Pages 8, 9.

1. $+12a$; $-9V$; $-4x$; $-4t^2$; $-29P^3$.
2. $-12ab$; $+2y^2$; $-5R$; $+4Ww$; $-5\pi R^2$.
3. $+n^2$; $-13n^2$; $-9n^2$; $-7n^2$; 0.

4. $10x$; $-14v$; $-8\,S$.
5. $-2x^2-11x+4$.
6. $-2a+4b-c$.
7. $4a^2+4a+1$.
8. $-6m+4n$.
9. $8x^3-4x^2+8x+5$.
10. $3r^2$.
11. $-3A^2B+3AB^2$.
12. $-2x^3-3x^2+4x-1$.
13. $4a-3a^2$; $5a^2-4a-2$.
14. $y-6z$ subtracted from $3y-z$; $2y+5z$.
15. $2T^2+6T-10$.
16. $2ab-2b^2$.

Page 10.

1. $3a$.
2. $2x-y$.
3. $-2n$.
4. $3t-5a$.
5. $5-x-x^2$.
6. M.
7. $6-2v$.
8. $2b^2-2ab$.
9. $-3W$.
10. $[4a-6b][-2a+2b]$.
11. $[P+Q-R][P-Q+R]$.
12. $[x+5y+a][x-5y-3a]$.
13. $(a^2-b^2)-(x^2+2xy+y^2)$.
14. $(2ab-a^2)-(m^2-n^2)$.
15. $(a^2-ab+b^2)-(x^2+2xy-y^2)$.
16. $(a^3-b^3)-(x^3-3x^2y+3xy^2-y^3)$.
17. $(5b-2a)-(3r+4s-t)$.
18. $(a^4-b^4)-(w^4-w^2v^2+v^4)$.

Page 12.

1. 720.
2. 480.
3. 840.
4. 840.
5. 3400.
6. 9600.
7. 700.
8. 90.
9. 1600.
10. 252.
11. $3\frac{1}{3}$.
12. 17,500.

Pages 12, 13.

1. $+12$; $+15$; -32; -14; $+27$; -35.
2. $+8$; -56; -30; $+36$.
3. -42; $+40$; $+36$; -135.
4. -6; -16; $+60$.
5. $+63$; $+64$; -125.
6. $+1$; $+360$.
7. $+4$; $+16$; $+9$; -1; $+8$.
8. -1; $+36$; $+100$; $+36$.
9. -8; $+20$; $+52$.
10. -65.

Pages 14, 15.

1. a^7; a^{10}; x^{10}; n^{16}; P^{11}.

2. y^6; t^{12}; D^{12}; m^{18}.

3. $15a^5$; $30V^6$; $18z^6$; $28b^{10}$.

4. $+12a^8$; $-40x^5$; $-12N^5$.

5. $-24a^4b^6$; $+18m^7n^7$; $+48x^6y^{10}$.

6. a^2-ab; $2m^2n+3mn^2$; $-4x^3y+12xy^3$.

7. $36n^3-60n^2$; $24A^6-32A^5+8A^4$; $x^4-x^3y+x^2y^2$.

8. $-15t^6+35t^5-10t^4$; $-3\pi R^4+6\pi R^3T-3\pi R^2T^2$.

9. $x^2-4x-21$; w^4-8w^2+15; $N^3-10N^2+28N-24$.

10. a^4-1; y^6-1; t^8-2t^4+1.

11. a^4+a^2+1.

12. $8s^4+26s^3-21s^2+26s-15$.

13. $-12x^5-3x^4+38x^3+23x^2-16x-40$.

14. $-4D^4-16D^3+19D^2+12D-12$.

15. n^5-1.

16. $p^3+6p^2+11p+6$.

17. A^4-16.

18. $81y^4-625$.

19. $v^{12}-x^{12}$.

20. $4a^2+a$.

21. $14x-47$.

22. $-xy-18x^2+35y^2$.

23. $2t^2-10$.

24. $2n^4-2n^3-4n^2-13n+5$.

25. $-2Q^2-22Q+10$.

26. $-2x^2y^2-2y^4$.

27. t^6; v^{20}; x^8.

28. $-125x^6y^9$; $36w^8v^{14}$; $-8H^{18}t^{15}$.

29. x^2+6x+9; $a^2-2ab+b^2$; $4a^2-12ab+9b^2$.

30. $9m^2+30mn+25n^2$; $a^2+b^2+c^2+d^2+2ab+2ac+2ad+2bc+2bd+2cd$.

Pages 16, 17.

1. 2^6.

2. 10^4.

3. 5^4.

4. 30^4.

5. 20^6.

6. $8a^6b^3$.

7. $9a^8b^4$.

8. $1000a^9b^{15}$.

9. $400x^2y^{12}$.

10. $27{,}000x^6y^{15}$.

11. $a^{2n}b^{3n}$.

12. $x^my^{2m}z^{3m}$.

13. $2^na^{2n}b^n$.

14. $a^{ncd}b^{mcd}$.

15. $64a^{18}b^6$.

16. n^2-16.

17. x^2-100.

18. R^2-1.

19. w^2-36.

20. $9-t^2$.

21. $4a^2-9b^2$.

22. $36r^2-s^2$.

23. $49m^2-4n^2$.

24. $x^2+7x+10$.

25. $T^2-11T+28$.

26. $V^2+5V-36$.

27. $y^2-13y+42$.

28. $4a^2+16a+15$.

29. $25x^2-20x-21$.

30. $144P^2+48P-45$.
31. $64a^2+72a+14$.
32. $x^2+2xy+y^2$.
33. $4n^2+20n+25$.
34. $R^2-14R+49$.
35. $25d^2-120d+144$.
36. $9x^2+24xy+16y^2$.
37. $225m^2-60mn+4n^2$.
38. $1-8t^2+16t^4$.
39. $81+36V+4V^2$.
40. $x^2+y^2+z^2+2xy-2xz-2yz$.
41. $16m^2+9n^2+p^2-24mn+8mp-6np$.
42. $1-4x+6x^2-4x^3+x^4$.
43. $16a^2+4b^2+25c^2+16ab-40ac-20bc$.
44. $a^4+a^3b-a^2b^2-ab^3+b^4$.
45. $x^2+y^2+z^2+w^2+2xy-2xz-2xw-2yz-2yw+2zw$.
46. $81+108t-64t^2$.
47. $9K^2-49L^2$.
48. $x^2y^2+2xy-143$.
49. $64M^4+16M^2-3$.
50. $256v^2t^2-s^4$.
51. $196A^6-1$.
52. 1681; 4896; 9991; 4225; 6804.
53. 11,009; 8464; 3584; 1575; 6241.

Pages 17, 18.

1. $+2$; $+6$; -2; -8; $+3$; -7.
2. $+6$; -7; -8.
3. a^5; t^5; V^6; R^{12}; n^8.
4. $8x^2$; $4P^4$; $16y^{12}$; $7T^3$.
5. $13a^5$; $-4V^4$.
6. $-5x^2y^3$; a^9.
7. $4x^2-2$; $-3m^2+2mn-4n^2$.
8. $-2ab+6b^2+3a^2$.
9. $-3r^3-4s^3+5r^2s$.
10. $8W^3+7W^2V-9WV^2+3V^3$.
11. $7x^2y^2-6y^4-8x^4$.
12. $M^2-\frac{1}{2}MN+N^2$.

Page 19.

1. $n+5$.
2. $x-4$.
3. $6+a$.
4. $t-6$.
5. $y+4$.
6. $3-v$.
7. $1-5N$.
8. $R+9$.
9. a^2+a+1.
10. $9-3m+m^2$.
11. $1+x+x^2+x^3$.
12. $s-t$.
13. A^2-AB+B^2.
14. $a^2-2ab-5a-10b+25+4b^2$.

Page 20.

1. x^2+5x+3, rem. 3.

2. $2x^3+8x^2+23x+95$, rem. 375.

3. a^3+2a^2+a+9, rem. 17.

4. $3n^2-2n+3$, rem. 6.

5. $y^3-9y^2+18y-34$, rem. 56.

6. t^2-2t+8, rem. 64.

7. $V^4+3V^3+9V^2+30V+91$, rem. 266.

8. $2r^2-5r+15$, rem. -52.

9. $A^3+A^2-4A+17$, rem. -76.

10. $n^3-6n^2+39n-234$, rem. 1395.

Page 21.

1. B^n.

2. $ttt\ldots\ldots$to a factors.

4. $(a-b+c)x$.

5. $(m+n-p)r^2$.

6. $(2a-3b+c)y$.

7. $(4a+9n-5b)x^2y$.

8. $(a+b+c)x+(b+c+a)y+(c+a+b)z$.

9. $5m-3n+4p$.

10. $(A-C)x-(B+D)y$.

11. $2ax-5by+(4c+a)z$.

12. -7.

13. $[5x+y-4z]\,[5x-3y+4z]$.

14. x^{8n}.

15. $a^{n+4}b^{n+2}$.

16. $-6R^{2a+2}$.

17. $A^{2x}-B^{2x}$.

18. $a^{4n}-b^{4n}$.

19. $T^{8a}-t^{8b}$.

20. x^{8n}.

21. x^{mn}.

22. $x^{na}y^{nb}$.

23. $2^ca^{nc}b^c$.

24. M^{3a}; B^{2x+2}.

25. $12x^{3m-n}y^{n-3m}$.

26. $8W^{a-4}V^2$.

27. $1-3N+2N^2$.

28. x^n+y^n.

29. $A^{3n}+A^{2n}B^n+A^nB^{2n}+B^{3n}$.

30. $3an^2$.

Pages 22, 23.

1. $6a^3(a-3)$.

2. $3mn(5m^2+4mn+3n^2)$.

3. $2x^3(2x^3-7xy^2+5y^3)$.

4. $2\pi R(R+2H)$.

5. $7t^2(t^2-4tv-2v^2)$.

6. $-A(A^2+A+1)$.

7. $-1(a^2+2ab+b^2)$.

8. $x^3y^3z^4(xz^2+x^2y+y^2z)$.

9. $4\pi(R^2+r^2+Rr)$.

10. $8V_1(V_1-2V_2+V_3)$.

11. $3m^2n(m^2-2mn+n^2)$.

12. $x^n(1+a)$.

13. $a^n(b^n-1)$.

14. $t^2v^2(v^n+t^n)$.

15. $P^n(P+Q)$.

16. $5a^{n-1}y^{n-1}(a+2y)$.

Page 24.

1. $(3A+4)(3A-4)$.
2. $(2t+5)(2t-5)$.
3. $(1+7n)(1-7n)$.
4. $(5x+8y)(5x-8y)$.
5. $(9V+1)(9V-1)$.
6. $(2P+3Q)(2P-3Q)$.
7. $(5R+4)(5R-4)$.
8. $(6v+7w)(6v-7w)$.
9. $(a-2b)(a^2+2ab+4b^2)$.
10. $(M-3)(M^2+3M+9)$.
11. $(1-4k)(1+4k+16k^2)$.
12. $(x-5y)(x^2+5xy+25y^2)$.
13. $(6-T)(36+6T+T^2)$.
14. $(m+3n)(m^2-3mn+9n^2)$.
15. $(A+1)(A^2-A+1)$.
16. $(2r+1)(4r^2-2r+1)$.
17. $(1+4P)(1-4P+16P^2)$.
18. $(5p+2q)(25p^2-10pq+4q^2)$.
19. $(2W-1)(4W^2+2W+1)$.
20. $(3+s)(9-3s+s^2)$.
21. $3(a+2b)(a-2b)$.
22. $5(1-2N)(1+2N+4N^2)$.
23. $6(2x+y)(4x^2-2xy+y^2)$.
24. $(x-1)(x+1)(x^2+1)$.
25. $(y+z)(y-z)(y^2-yz+z^2)(y^2+yz+z^2)$.
26. $(A-B)(A+B)(A^2+B^2)(A^4+B^4)$.
27. $(1+t)(1-t)(1+t+t^2)(1-t+t^2)$.
28. $(m^2+n^2)(m^4-m^2n^2+n^4)$.
29. $(x+y)(x-y)(x^2+y^2)(x^2-xy+y^2)(x^2+xy+y^2)(x^4-x^2y^2+y^4)$.
30. $a(m+n)(m-n)(m^2+n^2)$.
31. $p^3q^3(p+q)(p^2-pq+q^2)$.
32. $\pi R(R+H)(R-H)$.
33. $8vt(v-2t)(v^2+2vt+4t^2)$.
34. $(a+b+c)(a+b-c)$.
35. $(m-n+x)(m-n-x)$.
36. $(p+q-r)(p-q+r)$.
37. $(v+t+w+u)(v+t-w-u)$.
38. $(M-N+P-Q)(M-N-P+Q)$.
39. $(2A+B+C-3D)(2A+B-C+3D)$.

Page 26.

1. $(x-5)^2$.
2. $(2a+1)^2$.
3. $(3t-4v)^2$.
4. $(1+7P)^2$.
5. $(R-6r)^2$.
6. $(n+5)(n+2)$.
7. $(x-9)(x-2)$.
8. $(A-8)(A+7)$.
9. $(w+7v)(w-5v)$.
10. $(T-10U)(T+3U)$.
11. $(4a-3)(2a-1)$.
12. $(2V-5)(V+6)$.
13. $(5M-4)(3M+2)$.
14. $(4R+3r)(3R+4r)$.
15. $(2x+7y)(5x-3y)$.
16. $(8p+3q)^2$.
17. $(4W-9)(W+3)$.
18. $(L+7v)(L-2v)$.
19. $m(n-9)(n+6)$.
20. $5n(x+y^2)^2$.

21. $4a(F+5)(F-3)$.
22. $2n(2k-5m)(k+2m)$.
23. $-B(B+1)^2$.
24. $h^2(3m-2)(2m+7)$.
25. $(5r_1-2r_2)(r_1-2r_2)$.
26. $6(6+D)(1-D)$.
27. $t(T-15v)^2$.
28. $-4(1+3d^2)^2$.
29. $2\pi(2R-7)(R-2)$.
30. $3(5V_1+V_2)(3V_1+4V_2)$.
31. $(x+b-2)(x+b-3)$.
32. $(m-n+w+v)^2$.
33. $(4A+4B-9C)(A+B+3C)$.

Pages 27, 28.

1. $(a-b+c)(a-b-c)$.
2. $(x+y+1)(x-y-1)$.
3. $(a-b)(n-m)$.
4. $(y+x)(y-z)$.
5. $(x+y)(P-Q)$.
6. $(5A+B-C)(5A-B+C)$.
7. $(m-n+v+t)(m-n-v-t)$.
8. $(B+A+2C)(B-A-2C)$.
9. $(s-t)(P-5)$.
10. $(n+1)(a-b)$.
11. $(V-1)(V-v)$.
12. $(R-r)(R+r+1)$.
13. $(1+2y+z)(1+2y-z)$.
14. $(1-3b+m-2n)(1-3b-m+2n)$.
15. $(H-4t)(H+4t+v)$.
16. $(2+d)(a-b+c)$.
17. $(a+b)(m^2+m+1)$.
18. $(x-z+y-w)(x-z-y+w)$.

Page 28.

1. $(n^2+n+1)(n^2-n+1)$.
2. $(1+a+2a^2)(1-a+2a^2)$.
3. $(x^2+x+1)(x^2-x+1)(x^4-x^2+1)$.
4. $(t^2+2tv+3v^2)(t^2-2tv+3v^2)$.
5. $(v^2+3v+3)(v^2-3v+3)$.
6. $(2a^2+2ab+5b^2)(2a^2-2ab+5b^2)$.
7. $(2M^2+4MN+5N^2)(2M^2-4MN+5N^2)$.
8. $(R^2+3Rr-3r^2)(R^2-3Rr-3r^2)$.
9. $(6A^2+4AT-5T^2)(6A^2-4AT-5T^2)$.
10. $(7y^4+3y^2z+2z^2)(7y^4-3y^2z+2z^2)$.
11. $(2W^4+6W^2w^3+5w^6)(2W^4-6W^2w^3+5w^6)$.
12. $(8a^2b^2+2ab+1)(8a^2b^2-2ab+1)$.
13. $(m^4+4m^2xy^2-3x^2y^4)(m^4-4m^2xy^2-3x^2y^4)$.
14. $2\pi(2R^2+5Rr-5r^2)(2R^2-5Rr-5r^2)$.
15. $4t(v^6+3v^3t+4t^2)(v^6-3v^3t+4t^2)$.
16. $3a(5a^2+2ab^2c+2b^4c^2)(5a^2-2ab^2c+2b^4c^2)$.

Pages 30, 31.

1. $(x-1)(2x+1)$.
2. $(x+1)(3x+2)$.
3. $(2y-1)(y-2)$.
4. $(4n-1)(n+2)$.
5. $(t-3)(2t-3)$.
6. $(3A+5)(A+3)$.
7. $(V-5)(3V+2)$.
8. $(x+2)(x-2)(x^2+1)$.
9. $(a+1)(a+4)(a-5)$.
10. $(y-1)(y-2)(y-3)$.
11. $(b+2)(b^2-b-1)$.
12. $(P-1)(P+3)(P+5)$.
13. $(n+1)(n^3-2)$.
14. $(v-4)(v+6)(2v-1)$.
15. $(T-1)(3T^2+T+2)$.
16. $(a+2)(a^2+a+1)$.
17. $(R+3)(R^2-R-1)$.
18. $(s+3)(s+4)(s^2+1)$.
19. $(z-1)(z+1)(z^4+1)$.
20. $(a-b)(a+b)(a+2b)$.
21. $(x-y)(2x-y)$.
22. $(x+y)(3x-y)$.
23. $(m+y)(m-y)(m+2y)$.
24. $(s-2y)(s+3y)(s-3y)$.
25. $(t-d)(t+2d)(t+3d)$.
26. $(A-B)^3$.
27. $(N+a)(N+2a)(N-3a)$.
28. In $a^{20}-b^{20}$, let $a=b$.
29. In $a^{15}+b^{15}$, let $a=-b$.

Page 32.

1. $(a-b)(a^6+a^5b+a^4b^2+a^3b^3+a^2b^4+ab^5+b^6)$.
2. $(x+1)(x^6-x^5+x^4-x^3+x^2-x+1)$.
3. $(a-2b)(a^4+2a^3b+4a^2b^2+8ab^3+16b^4)$.
4. $(1-t)(1+t+t^2+t^3+t^4)$.
5. $(1-n)(1+n+n^2+n^3+n^4+n^5+n^6)$.
6. $(a-b)(a^{10}+a^9b+\dots\dots+ab^9+b^{10})$.
7. $(a+b)(a^{10}-a^9b+\dots\dots-ab^9+b^{10})$.
8. $(R+r)(R^{12}-R^{11}r+\dots\dots-Rr^{11}+r^{12})$.
9. $(R-r)(R^{12}+R^{11}r+\dots\dots+Rr^{11}+r^{12})$.
10. $(2x+3y)(16x^4-24x^3y+36x^2y^2-54xy^3+81y^4)$.
11. $2(1-2B)(1+2B+4B^2+8B^3+16B^4+32B^5+64B^6)$.
12. $(m^2+n)(m^8-m^6n+m^4n^2-m^2n^3+n^4)$.
13. $(x^2+y^2)(x^4-x^2y^2+y^4)$.
14. $(1+a^2)(1-a^2+a^4-a^6+a^8)$.
15. $(x^4+y^4)(x^8-x^4y^4+y^8)$.
16. $(n^2+1)(n^{12}-n^{10}+n^8-n^6+n^4-n^2+1)$.

17. $(v^2+w^2)(v^4-v^2w^2+w^4)(v^{12}-v^6w^6+w^{12})$.
18. $5(1+P^2)(1-P^2+P^4-P^6+P^8)$. 19. $4(1+N^2)(1-N^2+N^4)$.
20. $(a+b)(a^4-a^3b+a^2b^2-ab^3+b^4)(a^{10}-a^5b^5+b^{10})$.
21. $(1+y)(1-y+y^2-y^3+y^4)(1-y^5+y^{10}-y^{15}+y^{20})$.
22. $(a+1)(a-1)(a^2+a+1)(a^2-a+1)$.
23. $(x+y)(x-y)(x^2+xy+y^2)(x^2-xy+y^2)(x^2+y^2)(x^4-x^2y^2+y^4)$.
24. $(m+n)(m-n)(m^4+m^3n+m^2n^2+mn^3+n^4)(m^4-m^3n+m^2n^2-mn^3+n^4)$.
25. $(1-t)(1+t)(1+t^2)(1+t^4)(1+t^8)$. 26. $(16-a^3)(16+a^3)$.
27. $(a+1)(a-1)(a^6+a^5+a^4+a^3+a^2+a+1)(a^6-a^5+a^4-a^3+a^2-a+1)$.
28. $(p-q)(p+q)(p^2+q^2)(p^4+q^4)$.
29. $(w^2-v)(w^2+v)(w^4+w^2v+v^2)(w^4-w^2v+v^2)$.
30. $(1-P)(1+P+P^2)(1+P^3+P^6)$.

Pages 33, 34.

1. $(2N+1)(N-1)$.
2. $a(x^2+8a^2)(x^2-8a^2)$.
3. $ab(n+1)(3n-1)$.
4. $(2P+13Q)(2P+3Q)$.
5. $t(t-b)(5t+6b)$.
6. $(2V-W)(V-W)$.
7. $(x+1)(x-1)(x^2+4)$.
8. $(x-z)(y-z)$.
9. $(n+3)(n^2+6)$.
10. $(2a-2b+y+v)(2a-2b-y-v)$.
11. $(3R-4r)(3R-5r)$.
12. $(B^2+BC+C^2)(B^2-BC+C^2)$.
13. $(x^2+4xy-y^2)(x^2-4xy-y^2)$.
14. $(T^3+7)(T^3-6)$.
15. $(h^2+7k^2)(h^2-5k^2)$.
16. $y(y-3)(1+y)(1-y)$.
17. $(1-xt)(1+xt+bt)$.
18. $(x+y+z)(x+y-z)(x-y+z)(x-y-z)$.
19. $(V-v)(V+v)^3$.
20. $a(y+10)(y-3)$.
21. $(c-b)(a+d)$.
22. $(p+3)(p-2)(p^2-p+6)$.
23. $(8t+9)(9t-5)$.
24. $(a+1)(a-1)(b+1)(b-1)$.
25. $(2z-3t)(z+4t)$.
26. $-a^2(N-3a)^2$.
27. $(1+2m^2)(1-2m^2)(1+4m^4)$.
28. $(a-b)(a+b-3)$.
29. $n(c+n^2)(b+an)$.
30. $(a^2-b^2-c^2)^2$.
31. $(x-4a^4)(x-a^2b^2)$.
32. $(n+2x)(n-x-y)$.
33. $(2w^2+3wv+3v^2)(2w^2-3wv+3v^2)$.
34. $(A+C)(A-C)(B+C)$.
35. $(K-3)(K^2+2K+2)$.
36. $(n+3)(n^2-2n-1)$.

37. $(D-1)(D-3)(D+5)$.
38. $2(6+F)(36-6F+F^2)$.
39. $(1-x+a-b)(1-x-a+b)$.
40. $(P^2+P+1)(P^2-P+1)$.
41. $(ax+1)(x+b)$.
42. $(2t-b)(t-a)$.
43. $(2H+b)(H+2a)$.
44. $(z-1)(az+b)$.
45. $(x^2y^2+7a)(x^2y^2+2a)$.
46. $2(11+v)(10-v)$.
47. $5c(2c-3)^2$.
48. $(1+2Qn^2)(1-2Qn^2+4Q^2n^4-8Q^3n^6+16Q^4n^8)$.
49. $(1-d)(1+d+d^2)(1+d^3+d^6)$.
50. $(y-1)(y-3)(y-7)$.
51. $(L-1)(L-2)(L-3)$.
52. $(5x-y+3z)(5x-y-3z)$.
53. $(3a-3b-2)(a-b-4)$.
54. $(7a+7b+3c)(a+b-2c)$.
55. $(x+3)(x-4)(x^2-5)$.
56. $(a+1)(a-1)(a-A)$.
57. $b(m+2)(m-2)(2m+3)$.
58. $a(a+b)(a^2-ab+b^2)(a^6-a^3b^3+b^6)$.
59. $(v-t)(v+t)(v^2+t^2)(v^4+t^4)(v^8+t^8)$.
60. $(P^2+P-6Q)(P^2-P+6Q)$.
61. $(1-n)(1+n^2)(1-n^2+n^4)$.
62. $g^3(289g^4-83g^2+1)$.
63. $(x+y)(x^2-xy+y^2+1)$.
64. $(M-N)(1+M^2+MN+N^2)$.
65. $(1-a)(1+a+a^2)(1+a^3+a^6+a^9+a^{12}+a^{15}+a^{18})$.
66. $(x+2+2a-y)(x+2-2a+y)$.

Page 34.

1. 6.
2. 12.
3. 72.
4. $6a^3b^3$.
5. $2ma^2$.
6. $x-1$.
7. $V-1$.
8. $a+b$.
9. x^2+y.
10. $7N^2+5N+5$.
11. $p+q$.
12. $1+x+x^2$.
13. $R-5$.
14. $2v+2t$.
15. P^2+5P.
16. $m-2$.

Page 35.

1. 1512.
2. 1800.
3. 2016.
4. $144a^{10}b^5$.
5. $225\pi R^2r(R-r)(2R+r)^3$.
6. $(x+1)(x-1)^2$.
7. $2x^2y(x+y)(x^3-y^3)$.
8. $(N+1)(N-1)(N+2)(N^2+1)$.
9. $(m-n)(m+n)(m^2+mn+n^2)(m^2+n^2)$.

10. $1-V^8$. **12.** $(P^2+Q^2)^2(P^2-Q^2)^2$.
11. $6a^2(a-1)(a+3)(a^2+1)$. **13.** $(w+v)^2(w-v)^2$.
14. $(y+1)(y-1)(y+2)(y-2)(y+3)(y-3)$.
15. $(3T+2)(3T-2)^2(9T^2+6T+4)$.
16. $(6M^3-7M^2N-2\mathrm{MN}^2)(3M^2+MN-4N^2)$.

Pages 35, 36.

1. $(1+x)(1-x)^2$. **2.** $(a-b)(2a-2b+1)(2a-2b-1)$.
3. $(4c-1)(c^2-2)$. **4.** $(x-y)(x+y)(x^2+y^2)(x^4+y^4)(x^8+y^8)$.
5. $32(2^{x-1}-1)(2^{4x-4}+2^{3x-3}+2^{2x-2}+2^{x-1}+1)$.
6. $(xy-8)(x^2y^2+8xy+64)$. **13.** $(V^t-1)(V^{2t}+V^t+1)$.
7. $a^{2n}(a^n-b^n)$. **14.** $n(k^{5n}-n^{6n-1})$.
8. $(x^n-2)(x^n+1)$. **15.** $(a^n-b^n)(a^n+b^n-c^n)$.
9. $(t^{2n}+7v^n)(t^{2n}-4v^n)$. **16.** $(3b^n+5c^n)(2b^n-3c^n)$.
10. $(p^n+q^n)(p^n-q^n)$. **17.** $(x^{n-2}+y^{n+2})(x^{n-2}-y^{n+2})$.
11. $(A^n+B^n)(A^n-B^n)(A^{2n}+B^{2n})$. **18.** $(R^n+r)(\pi R^n+1)$.
12. $x(x^{2n}-y^n)$. **19.** $(V^{2t}+V^t+1)(V^{2t}-V^t+1)$.
20. $(N^a+1)(N^{4a}-N^{3a}+N^{2a}-N^a+1)$.
21. $(2^{2n}+1)^2$. **22.** $(a^2-b^2+3ab)(a^2-b^2-3ab)$. **23.** $2(2^{n-1}+1)(2^n+1)$.
24. $(a+b)(a^4-a^3b+a^2b^2-ab^3+b^4)(a^{10}-a^5b^5+b^{10})$.
25. $(3^n-5)(3^n+1)$. **31.** $2V^t-3$.
26. $(2^a+2^{2b})(2^a-2^{2b})$. **32.** $x+3$.
27. $(2^a-3^b)(2^{2a}+2^a3^b+3^{2b})$. **33.** $2a-3$.
28. $(6+a)(3-2a)$. **34.** $a-5$.
29. D^m-1. **35.** $x^{12}-y^{12}$.
30. $a^{2n}+b^n$. **36.** $(x+2)(x-2)(x-5)(x^2+4)$.
37. $(9by^3-3by^2-42by)(6b^2-26b+28)$.
38. $(a^n-b^n)(a^n+b^n)(a^{2n}+b^{2n})(a^{2n}+a^nb^n+b^{2n})$.
39. $(2P^x+1)^2(2P^x-1)^2$. **40.** $(x-y)^2(a^{n+1}-b^{n-1})$.

Page 38.

1. $-\frac{4}{7}$. **2.** $-\frac{5}{9}$. **3.** $\frac{2}{5}$. **4.** $-\frac{8}{9}$.
5. $\dfrac{2x^2}{3y^2}$. **6.** $-\dfrac{4ab}{5c^3}$. **7.** $-\dfrac{1}{t^3}$. **8.** $\dfrac{\pi r}{R}$.

9. $\dfrac{-1}{m-n}$. **10.** $\dfrac{mn}{m-n}$. **11.** $\dfrac{b-a}{m-n}$.

12. $\dfrac{3}{a-b}+\dfrac{4}{a-b}$. **15.** $\dfrac{6\pi r}{r^2-s^2}+\dfrac{r^2}{r^2-s^2}+\dfrac{r^2+s^2}{r^2-s^2}$.

13. $\dfrac{x}{x-1}-\dfrac{1}{x-1}+\dfrac{x+1}{x-1}$. **16.** $\dfrac{-1}{(a-b)(a-c)}$.

14. $\dfrac{1+2m}{m^2-1}+\dfrac{3m}{m^2-1}+\dfrac{m-2}{m^2-1}$. **17.** $\dfrac{a^3}{(a-b)(a-c)(a-d)}$.

18. $-\dfrac{a+b+c+d}{(a-b)(a-c)(a-d)}$.

Page 39.

1. $\frac{3}{4}$. **3.** $\frac{6}{7}$. **5.** $\frac{5}{7}$. **7.** $\dfrac{xy}{z^2}$. **9.** $\dfrac{2a}{3b}$.

2. $\frac{1}{3}$. **4.** $\frac{8}{9}$. **6.** $\frac{1}{14}$. **8.** $\dfrac{1}{2A^2}$. **10.** $\dfrac{1}{2R^2}$.

11. $\dfrac{x+1}{x^2+x+1}$. **14.** $\dfrac{t-7}{t+6}$. **17.** $\dfrac{w-v}{wv}$.

12. $\dfrac{a^2-ab+b^2}{a^2}$. **15.** $\dfrac{5p+q}{4p+5q}$. **18.** $\dfrac{F^4-F^3+F^2-F+1}{(F+1)^4}$.

13. $\dfrac{m^2-n^2}{m^2+n^2}$. **16.** $\dfrac{1-N+N^2}{N^2-5N}$. **19.** $\dfrac{1}{A^2-AB+B^2}$.

20. $\dfrac{a+x}{a-y}$.

Page 40.

1. $\frac{10}{12}, \frac{9}{12}, \frac{6}{12}$. **3.** $\frac{12}{42}, \frac{21}{42}, \frac{28}{42}$. **5.** $\frac{30}{225}, \frac{72}{225}, \frac{35}{225}$. **7.** $\frac{42}{84}, \frac{64}{84}, \frac{75}{84}$.

2. $\frac{8}{24}, \frac{6}{24}, \frac{9}{24}$. **4.** $\frac{20}{48}, \frac{9}{48}, \frac{14}{48}$. **6.** $\frac{27}{216}, \frac{30}{216}, \frac{160}{216}$. **8.** $\frac{405}{720}, \frac{160}{720}, \frac{192}{720}$.

9. $\dfrac{80a}{60a^3}, \dfrac{12a^2}{60a^3}, \dfrac{35}{60a^3}$. **12.** $\dfrac{1+x}{1-x^2}, \dfrac{1-x}{1-x^2}, \dfrac{x}{1-x^2}$.

10. $\dfrac{a^3c^2}{a^2b^2c^2}, \dfrac{a^2b^5}{a^2b^2c^2}, \dfrac{b^2c^3}{a^2b^2c^2}$. **13.** $\dfrac{a(a-b)(a^2+b^2)}{a^4-b^4}, \dfrac{b(a+b)(a^2+b^2)}{a^4-b^4}, \dfrac{ab(a+b)(a-b)}{a^4-b^4}$.

11. $\dfrac{m^2}{mnp}, \dfrac{p^2}{mnp}, \dfrac{n^2}{mnp}$. **14.** $\dfrac{M^3-MN^2}{M^2N-N^3}, \dfrac{M^2+M^2N}{M^2N-N^3}, \dfrac{MN^2}{M^2N-N^3}$.

15. $\dfrac{t^2-2t}{(t+3)(t-3)(t-2)}, \dfrac{t^2+2t-3}{(t+3)(t-3)(t-2)}$.

16. $\dfrac{R^2-R-6}{(R-7)(R-2)(R+2)}$, $\dfrac{-3R+21}{(R-7)(R-2)(R+2)}$.

17. $\dfrac{m^2n+mn^2}{mn(m^3-n^3)}$, $\dfrac{2m^2n+2mn^2+2n^3}{mn(m^3-n^3)}$.

18. $\dfrac{6x+3}{(x+1)(x+2)(2x+1)}$, $\dfrac{5x+5}{(x+1)(x+2)(2x+1)}$, $\dfrac{7x+14}{(x+1)(x+2)(2x+1)}$.

19. $\dfrac{b-c}{(a-b)(a-c)(b-c)}$, $\dfrac{c-a}{(a-b)(a-c)(b-c)}$, $\dfrac{a-b}{(a-b)(a-c)(b-c)}$.

20. $\dfrac{y^2-z^2}{(x-y)(x-z)(y-z)}$, $\dfrac{z^2-x^2}{(x-y)(x-z)(y-z)}$, $\dfrac{x^2-y^2}{(x-y)(x-z)(y-z)}$.

Pages 41, 42.

1. $\frac{31}{20}$. 4. $\frac{257}{240}$. 7. $\dfrac{a^2+b^2+c^2}{abc}$.

2. $\frac{17}{48}$. 5. $\dfrac{6+a}{2a^2}$. 8. $\dfrac{4N}{N^2-1}$.

3. $\frac{97}{168}$. 6. $\dfrac{x-2}{2x^2}$. 9. $\dfrac{1+2t}{1-t^2}$.

10. $\dfrac{7s+3}{s^3-s}$. 12. $\dfrac{4RH}{H^2-R^2}$.

11. $\dfrac{2r^2+14r-6}{(r+2)(r+7)(r+8)}$. 13. $\dfrac{V^2+18}{V^2-9}$.

14. $\dfrac{2a^2+ab+b^2}{a^3-b^3}$. 15. $\dfrac{1}{(a-b)(a-c)}$. 16. 0.

17. $\dfrac{M^2P-M^2N+MN^2-PN^2+P^3-NP^2}{(M-N)(M-P)(P-N)}$.

18. $\dfrac{24r^3-8rR^2}{(r^2-R^2)(9r^2-R^2)}$. 23. $\dfrac{3V+6}{V+3}$.

19. $\dfrac{18n^2-360}{(n^2-16)(n^2-25)}$. 24. $\dfrac{ab}{a-b}$.

20. $\dfrac{4bc^2-4bd^2+8a^2d-8b^2d}{(a^2-b^2)(c^2-d^2)}$. 25. $\dfrac{1}{1+R}$.

21. $\dfrac{16x^4-1476}{(x^4-81)(x^4-1)}$. 26. $\dfrac{m^2+n^2}{m+n}$.

22. $\dfrac{t^2-1}{t}$. 27. $\dfrac{5+2y}{4-y}$.

28. $\frac{N^2+MN+M^2}{N^2}$.
29. $\frac{x^3}{x-y}$.
30. 0.
31. $\frac{2D^4-D^2+1}{D^2-D+1}$.

Pages 43, 44.

1. $\frac{3}{10}$.
2. $\frac{4}{7}$.
3. 4.
4. 8.
5. $\frac{1}{12}$.
6. $\frac{17}{32}$.
7. $\frac{19}{12}$.
8. $\frac{5}{3}$.
9. $\frac{ad}{b}$.
10. 1.
11. m^3n.
12. x^3-x^2+x-1.
13. $\frac{v^2-2t^2}{v^3+3v^2t+3vt^2+2t^3}$.
14. $\frac{n+1}{n+5}$.
15. $\frac{RH}{R^2-2RH+H^2}$.
16. b^2-2b+1.
17. $\frac{V^2+3VT+9T^2}{V^3}$.
18. $\frac{1}{A-1}$.
19. $a+b$.
20. $\frac{1-N^4}{N^4}$.
21. $\frac{1}{F^3-F^2}$.
22. $\frac{a+b}{a^2}$.
23. $\frac{mn^2-m^3}{n^3}$.
24. $\frac{h}{h-k}$.
25. $\frac{-2P^3+2P^2+P-1}{P^2+P^3}$.
26. $2n$.
27. B.
28. x^2-y^2.
29. $2K^2-2$.
30. $-v^2-2$.
31. $4M^2-11M+2$.

Page 45.

1. $\frac{7}{8}$.
2. $\frac{25}{24}$.
3. 20.
4. $\frac{3}{32}$.
5. 27.
6. $\frac{22}{5}$.
7. $\frac{8}{45}$.
8. $\frac{4}{15}$.
9. $\frac{2m^2}{15pn}$.
10. $\frac{2R}{3}$.
11. x^2-x.
12. $\frac{6-V-V^2}{2}$.
13. $\frac{a^2-ab+b^2}{a^2}$.
14. $\frac{t-1}{t}$.
15. $\frac{3m^3-m^2n-mn^2-4n^3}{12m^2+11mn-5n^2}$.
16. $\frac{P^3+P^2Q+PQ^2}{Q^3}$.
17. $\frac{b}{a^2-ab}$.
18. $\frac{x^2+2}{x^2+x-3}$.
19. $\frac{M^3-M^2N+MN^2}{M-N}$.
20. $\frac{t^2-tk}{t^2+tk+k^2}$.

21. $\dfrac{r_1r_2r_3}{r_1r_2+r_1r_3+r_2r_3}$.

Pages 46, 47.

1. $\frac{14}{9}$.
2. $\frac{48}{5}$.
3. $\frac{4}{5}$.
4. $\dfrac{N^2}{M^2}$.
5. $\dfrac{x^2}{y^2}$.
6. $\dfrac{x^2}{1+x}$.
7. $\dfrac{A-B}{A+B}$.
8. $\dfrac{n-m}{n+m}$.
9. $\dfrac{b-c}{c}$.
10. $\dfrac{-b}{a^2+2ab+b^2}$.
11. $t+s$.
12. $\dfrac{a^2+b^2}{a^2-b^2}$.
13. $\dfrac{mn}{m^2-mn+n^2}$.
14. $\dfrac{2}{x+1}$.
15. $\dfrac{a^2-a+1}{2a-1}$.
16. $\dfrac{3x+2}{4x+3}$.
17. $\dfrac{nE}{nR+r}$.
18. 2.98−.

Pages 47, 48.

1. $\dfrac{2}{t}$.
2. $\dfrac{3V}{2+2V}$.
3. −1.
4. $\dfrac{(x+y+z)^2}{x^2+xy-xz}$.
5. $\dfrac{b^4+5a^2b+5ab^2-25a^4}{5ab^2}$.
6. 0.
7. 1.
8. $\dfrac{a^3-5a^2n-5an^2+n^3+a^3n-2a^2n^2+an^3}{an(a-n)(a+n)}$.
9. $w+v$.
10. $\dfrac{a^2-x^2}{a^2+x^2}$.
11. 1.
12. $\dfrac{-1}{x}$.
13. $\dfrac{1}{a}$.
14. x.
15. $\dfrac{1}{m^2-n^2}$.
16. $\dfrac{a^6+b^6}{a^6+2a^3b^3+b^6}$.
17. $\dfrac{ab}{a^2+b^2}$.
18. $\dfrac{(A-B)^2}{(A+B)^2}$.

Page 51.

1. $n=3$.
2. $t=10$.
3. $V=6$.
4. $x=1$.
5. $P=-2$.
6. $k=2$.
7. $N=8$.
8. $a=24$.
9. $x=\frac{12}{11}$.
10. $H=-\frac{12}{5}$.
11. $m=4\frac{68}{165}$.

Page 53.

1. $n=6.$
2. $V=1.$
3. $x=3.$
4. $A=\frac{17}{18}.$
5. $R=\frac{1}{4}.$
6. $t=1.$
7. $N=\frac{1}{2}.$
8. $y=-27.$
9. $P=-3.$
10. $a=1\frac{5}{8}.$
11. $r=-8.$
12. $x=6.$
13. $K=9.$
14. $b=10.$
15. $y=2\frac{10}{23}.$
16. $B=-\frac{1}{4}.$

Pages 54, 55.

1. $a=4.$
2. $x=2.$
3. $m=8.$
4. $t=5.$
5. $b=2.$
6. $V=-\frac{4}{5}.$
7. $R=1.$
8. $t=4.$
9. $a=-3.$
10. $k=-3\frac{9}{10}.$
11. $x=2\frac{1}{2}.$
12. $n=11.$
13. $W=-13.$
14. $x=0.$

Pages 55, 56.

1. $x=\dfrac{m^2-n^2}{5m}.$

2. $n=\dfrac{2P}{1-P}.$

3. $V=\dfrac{1+d}{2d}.$

4. $x=\dfrac{na-y}{n}.$

5. $A=\dfrac{BC}{B-C+BC}.$

6. $t=\dfrac{12a+12b}{3a+b}.$

7. $b=\dfrac{2A-hb'}{h}.$

8. $R=\dfrac{S-\pi Lr}{\pi L};$

$r=\dfrac{S-\pi LR}{\pi L};$

$L=\dfrac{S}{\pi(R+r)}.$

9. $L=\dfrac{S-\pi(R^2+r^2)}{\pi(R+r)}.$

10. $H=\dfrac{3V}{\pi(R^2+r^2+Rr)}.$

11. $F=\dfrac{Dd+df+Df}{f+D};$

$f=\dfrac{Dd-DF}{F-d-D};$

$D=\dfrac{Ff-df}{f-F+d};$

$d=\dfrac{Ff-Df+DF}{D+f}.$

12. $R=\dfrac{r_1r_2r_3r_4r_5}{r_2r_3r_4r_5+r_1r_3r_4r_5+r_1r_2r_4r_5+r_1r_2r_3r_5+r_1r_2r_3r_4};$ etc.

13. $K=\dfrac{H}{ND(D-1)(R+2)};\quad N=\dfrac{H}{KD(D-1)(R+2)};$

$R=\dfrac{H-2KND(D-1)}{KND(D-1)}.$

14. $H=\dfrac{P(460+T)(460+t)}{7.6t-7.9T-138};\quad T=\dfrac{H(7.6t-138)-460P(460+t)}{460P+Pt+7.9H};$

$$t=\frac{138H+211600P+460PT+7.9HT}{7.6H-PT-460P}.$$

15. $V=\frac{V'P'(273+t)}{P(273+t')}$; $V'=\frac{VP(273+t')}{P'(273+t)}$; $P=\frac{V'P'(273+t)}{V(273+t')}$;

$P'=\frac{VP(273+t')}{V'(273+t)}$; $t=\frac{273VP+VPt'-273V'P'}{V'P'}$:

$t'=\frac{273V'P'+V'P't-273VP}{VP}$.

Pages 57–61.

1. 24, 26.
2. $37\frac{1}{2}$ lb. of 20¢ grade;
   $62\frac{1}{2}$ lb. of 28¢ grade.
3. 3 lb. of 24¢ grade;
   6 lb. of 30¢ grade.
4. $6\frac{2}{3}$ lb. of 50¢ grade;
   $3\frac{1}{3}$ lb. of 80¢ grade.
5. 18 nickels; 15 dimes.
6. $3500 at 6%; $6500 at 5%.
7. $16,000 at 4%; $9000 at $5\frac{1}{2}$%
8. $4000 at 4%; $8000 at 8%.
9. $5\frac{5}{11}$ gal. cream; $54\frac{6}{11}$ gal. milk.
10. $17\frac{21}{47}$ gal. cream; $2\frac{26}{47}$ gal. milk.
11. 25 gal.
12. $3\frac{3}{4}$ qt.
13. 2 lb.
14. 30 lb. tin; 20 lb. zinc.
15. 24 yr.
16. $7.20.
17. $2.50.
18. $26\frac{2}{3}$ in.; $21\frac{1}{3}$ in.
19. 6 hr.; 36 mi.
20. 4 hr.
21. 1650 ft.
22. $10\frac{10}{11}$ min. after 2.
23. $49\frac{1}{11}$ min. after 3.
24. $10\frac{10}{11}$ min. after 8.
25. $54\frac{6}{11}$ min. after 7.
26. $65\frac{5}{11}$ min.
27. $586\frac{17}{28}$ days.
28. Dec. 29, 1913.
29. $115\frac{265}{277}$ days.
30. $398\frac{2281}{3965}$ days; April 3, 1910.
31. $3\frac{3}{5}$ days.
32. $5\frac{5}{13}$ days.
33. $13\frac{1}{3}$ hr.
34. $8\frac{2}{11}$ hr.
35. 25 mi. per hour.
36. 15 mi. per hour.
37. $1\frac{19}{21}$ mi. per hour.
38. $\frac{pqr}{pq+q-p}$.

Pages 62, 63.

1. $n=1$.
2. $A=3$.
3. $x=3\frac{1}{2}$.
4. $x=7$.
5. $t=4$.
6. $x=\frac{bc}{c^2-b}$.
7. $n=\frac{5b}{3}$.
8. $V=\frac{2TE-2RTE}{3pR-5p}$; $p=\frac{2TE-2RTE}{3RV-5V}$; $E=\frac{5pV-3pRV}{2RT-2T}$;

$$T=\frac{5pV-3pRV}{2RE-2E}.$$

9. 7 hr.
10. 82.
11. 30 mi. per hour.
12. 4.
13. 1:18 P.M. the next day; 1129.6 mi.
15. $\frac{19}{37}$.
16. $\frac{4}{55}$.
17. $\frac{4}{33}$.
18. $\frac{2}{33}$.
19. $\frac{41}{333}$.
20. $\frac{9}{11}$.
21. $\frac{16}{37}$.
22. $\frac{58}{111}$.
23. $\frac{2057}{3333}$.
24. $1\frac{2}{3}$.

Page 66.

12. $4\frac{4}{9}$° C.; $-11\frac{1}{9}$° C.; $-7\frac{7}{9}$° C.; $-23\frac{1}{3}$° C.; $-28\frac{8}{9}$° C.; $-20\frac{5}{9}$° C.; $93\frac{1}{5}$° F.; 59° F.; 104° F.; $46\frac{2}{5}$° F.; 14° F.

Page 67.

1. $x=6,\ y=2$.
2. $x=4\frac{6}{13},\ y=1\frac{5}{13}$.
3. $x=3,\ y=2$.
4. $x=2,\ y=3$.
5. $x=3,\ y=-5$.
6. $x=2,\ y=3$.
7. $x=9,\ y=2$.
8. $x=5,\ y=7$.
9. $x=17,\ y=13$.
10. Parallel.

Pages 69, 70.

1. $m=5,\ n=7$.
2. $A=4,\ B=3$.
3. $x=5,\ y=1$.
4. $v=6\frac{7}{13},\ t=-1\frac{2}{13}$.
5. $R=-2,\ r=-3$.
6. $V=5,\ t=6$.
7. $p=13,\ q=17$.
8. $M=4,\ N=-5$.
9. $x=2\frac{12}{17},\ y=7\frac{2}{17}$.
10. $a=2,\ b=1$.
11. $E=7,\ F=9$.
12. $m=15,\ n=8$.
13. $a=17,\ b=13$.
14. $w=9,\ v=5$.
15. $r=-11,\ r'=7$.
16. $F=1,\ C=-1\frac{1}{2}$.
17. $A=12,\ B=-3$.
18. $y=5,\ z=7$.
19. $D=1\frac{57}{59},\ T=\frac{5}{59}$.
20. $k=\frac{1}{3},\ s=\frac{1}{4}$.
21. $x=-1,\ y=3$.
22. $a=20,\ b=32$.
23. $a=-7\frac{1}{5},\ b=-17\frac{3}{5}$.
24. $m=10,\ n=24$.
25. $x=a-b,\ y=b-a$.
26. $x=a,\ y=b$.
27. $x=a,\ y=-\dfrac{b}{a}$.
28. $x=\dfrac{m^2+n^2}{m+2n},\ y=\dfrac{2m-n}{m+2n}$.
29. $x=\dfrac{p^2}{p^2+q^2},\ y=\dfrac{-1}{p^2+q^2}$.
30. $x=\dfrac{ce-bf}{ae-bd},\ y=\dfrac{af-cd}{ae-bd}$.
31. $x=\frac{1}{13},\ y=\frac{1}{17}$.
32. $a=-\frac{1}{2},\ b=-\frac{1}{3}$.
33. $A=\frac{1}{3},\ B=\frac{1}{7}$.
34. $m=-1,\ n=\frac{1}{3}$.
35. $v=\frac{1}{4},\ t=-\frac{1}{5}$.
36. $M=\frac{1}{8},\ N=\frac{1}{2}$.

Page 72.

1. $a=1,\ b=2,\ c=3$.
2. $x=2,\ y=3,\ z=4$.
3. $M=0,\ N=-4\frac{21}{89},\ P=-\frac{80}{89}$.
4. $r=7\frac{3}{13},\ s=3\frac{5}{13},\ t=0$.

5. $A=2, B=0, C=0$.
6. $x=-7, y=8, z=1$.
7. $a=4, b=5, c=6$.
8. $x=1, y=2, z=3, w=4$.
9. $p=2\frac{10}{13}, q=\frac{2}{13}, r=-1\frac{11}{13}, s=5\frac{10}{13}$.
10. $E=\frac{1}{2}, G=0, H=4, K=-2$.
11. $a=\frac{1}{3}, b=\frac{1}{2}, c=1$.
12. $x=\frac{1}{12}, y=\frac{1}{7}, z=\frac{1}{5}$.
13. $x=a+b, y=a-b, z=2a$.
14. See § 60 for solution.

Page 74.

1. $x=5, y=2$.
2. All three meet at one point.
3. By multiplication by 3 and transposition.
4. They are identical.
6. They are parallel.

Pages 74–78.

1. A, $2400; B, $1200.
2. $6400 at 5%; $3600 at 8%.
3. $15,000 in bonds; $9000 in stock.
4. 30 lb. of 20¢ grade; 20 lb. of 30¢ grade.
5. 12 lb. of 40¢ grade; 8 lb. of 65¢ grade.
6. $35\frac{5}{9}$ gal. cream; $4\frac{4}{9}$ gal. milk.
7. $97\frac{23}{41}$ gal. milk; $2\frac{18}{41}$ gal. cream.
8. $46\frac{2}{13}$ qt. mixture; $13\frac{11}{13}$ qt. alcohol.
9. 90%; 80%.
10. Gold, 11 oz.; silver, 7 oz.
11. Gold, $15\frac{1}{8}$ lb.; silver, $4\frac{7}{8}$ lb.
12. Tin, 30 lb.; zinc, 15 lb.
13. Fahrenheit, 176°; Centigrade, 80°.
14. Fahrenheit, 662°; Centigrade, 350°.
15. 126 lb.; 90 lb.
16. $33\frac{1}{3}$ lb.; $26\frac{2}{3}$ lb.
17. 15 mi. per hour; 30 mi. per hour.
18. Passenger, 66 ft. per second; freight, 44 ft. per second.
19. Man, 40 yr.; son, 10 yr.
20. 20 hr.; 16 hr.
21. 30 hr.; 24 hr.
22. 9 in.; 12 in.; 15 in.
23. 12 in.; 16 in.; 22 in.
24. 4 in.; 8 in.; 12 in.
25. Sides are divided into 12 in. and 16 in., 12 in. and 24 in., 16 in. and 24 in.
26. 8 hr.; 9 hr.; 12 hr.

Pages 78, 79.

1. 26.
2. -35.
3. -1.
4. 36.
5. -10.
6. -10.
7. 16.
8. 10.
9. -30.
10. $ad-bc$.
11. $2x$.
12. a^2-1.

Page 80.

1. $x=1, y=2.$
2. $M=2, N=3.$
3. $p=3, q=1.$
4. $a=1, b=4.$
5. $R=5, T=4.$
6. $v=\frac{86}{89}, t=1\frac{58}{89}.$
7. $x=\frac{29}{33}, y=1\frac{5}{33}.$
8. $h=\frac{7}{9}, k=1\frac{8}{9}.$
9. $P=1\frac{3}{11}, Q=1\frac{6}{11}.$
10. $m=4, n=4.$
11. $a=-\frac{1}{12}, V=-\frac{17}{24}.$
12. $m=\frac{1}{6}, n=2\frac{2}{3}.$

Page 81.

1. $-22.$
2. $-107.$
3. $177.$
4. $4.$
5. $0.$
6. $70.$
7. $10.$
8. $250.$
9. $268.$
10. $a-2b+c.$
11. $x^2+2yz-2xz-y^2.$
12. $1-n^2.$

Page 83.

1. $x=4, y=1, z=2.$
2. $A=5, B=2, C=1.$
3. $p=1, q=-1, r=4.$
4. $M=-2, N=3, P=-1.$
5. $a=10, b=-6, c=1.$
6. $r=\frac{22}{23}, s=1\frac{57}{115}, t=\frac{51}{115}$
7. $x=3, y=-4\frac{1}{2}, z=5\frac{1}{2}.$
8. $m=\frac{9}{25}, n=\frac{18}{25}, k=\frac{23}{25}.$
9. $A=6, B=4, C=2.$
10. Indeterminate.

Pages 83, 84.

1. $a=1\frac{121}{139}, b=7\frac{9}{73}.$
2. $m=\frac{1}{5}, n=\frac{1}{4}.$
3. $x=2, y=3.$
4. $a=5, b=7.$
5. $m=\frac{1}{2}, n=\frac{1}{2}.$
6. $A=7, B=9, C=6.$
7. $x=2, y=1, z=1.$
8. $\frac{P}{Q}=\frac{ab}{a'b'}.$
9. $V=\frac{1}{3}H(B+b+\sqrt{Bb}).$
10. $P_1=\frac{wd_2}{d_1+d_2}, P_2=\frac{wd_1}{d_1+d_2}.$
11. $x=\frac{b+c}{2}, y=\frac{a-c}{2}, z=\frac{a-b}{2}.$
12. $x=\frac{bdr+cpf-bqf}{acf+bde}, y=\frac{pde-adr+aqf}{acf+bde}, z=\frac{acr-cpe+bqe}{acf+bde}.$
13. $x=\frac{2}{5}, y=\frac{2}{3}, z=2.$
14. $x=\frac{2}{a+b-c}, y=\frac{2}{a-b+c}, z=\frac{2}{-a+b+c}.$

Page 87.

1. 5.
2. 9.
3. n^3.
4. P^5.
5. $4ab^2$.
6. $6V^2t^3$.
7. $10m^2n^4$.
8. $11AB^6$.
9. $2p^3qr^2$.
10. $3x^2y^3z^4$.
11. 3.
12. 4.
13. A^2.
14. s^4.
15. $2xy^3$.
16. $3P^2Q^3$.
17. $5a^4b^5$.
18. $4W^2V^6$.
19. $2xy^2z^3$.
20. $3m^4np^7$.
21. 2.
22. 3.
23. a^2b^3.
24. $4xy^4$.
25. $5M^2N^3P$.
26. 2.
27. 3.
28. xy^2.
29. $2A^3B^4$.
30. $p^5q^2r^6s^4$.

Pages 88, 89.

1. $A-3$.
2. $2t+3$.
3. $1-5V$.
4. $7m^2+2f$.
5. $6a-4b$.
6. $8x+y$.
7. n^2-3n+2.
8. $1+5k+2k^2$.
9. t^3-5t-2.
10. $5R^3-4R^2+3$.
11. a^2-ab+b^2.
12. $6a+b-2c$.
13. $x-5y+3z$.
14. v^3-2v^2+5v-2.
15. $1-2x-2x^2-4x^3$.
16. $1+4y-8y^2+32y^3$.
17. $1-\frac{1}{2}a-\frac{1}{8}a^2-\frac{1}{16}a^3$.
18. $x-3$.
19. $m-2n$.

Pages 90, 91.

1. 47.
2. 92.
3. 362.
4. 806.
5. 1635.
6. 8.3.
7. 1.27.
8. 0.43.
9. 0.086.
10. 3.464.
11. 2.645.
12. 4.472.
13. 0.447.
14. 1.268.
15. 0.223.
16. 80 inches.
17. 11.49− inches.
18. 21.95+ inches.
19. 4.37+ inches.
20. 29.39+ inches.
21. 10.39+ inches.
22. 94.10+ sq. in.
23. 322.49+ sq. in.

Page 92.

1. 3.2.
2. 3.9.
3. 4.5.
4. 5.3.
5. 6.3.
6. 7.2.
7. 7.7.
8. 2.5.
9. 1.5.
10. 3.5.
12. 2.3; 2.7; 3.6; 4.6; 5.3.

Pages 95, 96.

1. $2\sqrt{2}$.
2. $3\sqrt{3}$.
3. $2\sqrt{3}$.
4. $6\sqrt{2}$.
5. $2\sqrt[3]{2}$.
6. $2\sqrt[3]{3}$.
7. $2\sqrt[3]{9}$.
8. $5\sqrt[3]{2}$.
9. $2\sqrt[4]{2}$.
10. $3\sqrt[4]{2}$.
11. $2\sqrt[5]{2}$.
12. $2\sqrt[5]{7}$.
13. $\frac{1}{3}\sqrt{3}$.
14. $\frac{1}{5}\sqrt{10}$.
15. $\frac{2}{3}\sqrt{6}$.
16. $\frac{1}{2}\sqrt[3]{4}$.
17. $\frac{1}{2}\sqrt[3]{6}$.
18. $\frac{1}{7}\sqrt[3]{245}$.
19. $\frac{1}{2}\sqrt{2}$.
20. $\frac{1}{2}\sqrt[4]{6}$.
21. $ab\sqrt{a}$.
22. $m^2\sqrt{3mn}$.
23. $2xy^3\sqrt{2y}$.
24. $v\sqrt[3]{5vt}$.
25. $2Rr\sqrt[3]{2r}$.
26. $2B\sqrt[4]{2A^2B}$.
27. $4x^3y\sqrt{5}$.
28. $\sqrt{ab}$.
29. $b\sqrt[3]{a^2b}$.
30. $xy\sqrt{2x}$.
31. $wv\sqrt{3v}$.
32. $Q\sqrt{2PQ}$.
33. $2\sqrt[12]{16x^9y^6}$.
34. $\frac{1}{b}\sqrt{ab}$.
35. $\frac{1}{y^2}\sqrt{xy}$.
36. $\frac{1}{2n}\sqrt{6mn}$.
37. $\frac{1}{2x}\sqrt{2x}$.
38. $\frac{1}{5c^2}\sqrt[3]{25a^2bc^2}$.
39. $\frac{2x}{3y}\sqrt[3]{2x^2y}$.
40. $\frac{v}{7t}\sqrt{35vt}$.
41. $\frac{1}{a-b}\sqrt{a^2-b^2}$.
42. $\frac{m}{m+n}\sqrt{mn+n^2}$.
43. $\frac{1}{2A}\sqrt{B^2-4AC}$.
44. $\frac{1}{x^2y^2}\sqrt{x^5y+xy^5}$.
45. $\frac{9}{2x}\sqrt[3]{x^2y}$.
46. $\frac{a^3b}{6}\sqrt[5]{9ab^2}$.
47. $\frac{1}{K^3}\sqrt{K-K^2}$.
48. $2\sqrt{1-2\sqrt{3}}$.
49. $R\sqrt{1-3\sqrt{5}}$.
50. $(a-b)\sqrt{a+b}$.

Page 97.

1. $5\sqrt{2}$.
2. $2\sqrt{3}$.
3. $\sqrt{5}$.
4. $14\sqrt{6}$.
5. $\sqrt[3]{2}$.
6. $24\sqrt[3]{3}$.
7. $\frac{3}{4}\sqrt{2}$.
8. $\frac{1}{12}\sqrt{6}$.
9. $\frac{5}{6}\sqrt[3]{6}$.
10. $\frac{7}{2}\sqrt[3]{12}$.
11. $7\sqrt{2}$.
12. $3\sqrt{3}$.
13. $4\sqrt{n}$.
14. $20\sqrt[3]{t^2}$.
15. $a\sqrt{a}$.
16. $6m\sqrt{m-n}$.
17. $7m\sqrt{m-n}$.
18. $3A\sqrt[3]{A+B}$.
19. $\frac{2}{x}\sqrt{5xy}$.
20. $10\sqrt{p^2-q^2}$.

Page 98.

1. $\sqrt[12]{n^8}$, $\sqrt[12]{n^9}$, $\sqrt[12]{n^{10}}$.
2. $\sqrt[20]{V^{10}}$, $\sqrt[20]{V^{10}}$, $\sqrt[20]{V^{12}}$.
3. $\sqrt[6]{8x^3}$, $\sqrt[6]{9x^2}$, $\sqrt[6]{4x^4}$.
4. $\sqrt[12]{729t^{18}}$, $\sqrt[12]{16t^8}$, $\sqrt[12]{t^3}$.
5. $\sqrt[20]{a^{50}b^{30}}$, $\sqrt[20]{a^{16}b^4}$, $\sqrt[20]{a^5b^{15}}$.
6. $\sqrt[6]{8}$, $\sqrt[6]{9}$.
7. $\sqrt[12]{256}$, $\sqrt[12]{216}$, $\sqrt[12]{64}$.
8. $\sqrt[30]{729N^{18}}$, $\sqrt[30]{32N^{10}}$.
9. $\sqrt[12]{\frac{1}{64}}$, $\sqrt[12]{\frac{1}{81}}$, $\sqrt[12]{\frac{1}{64}}$.
10. $\sqrt[12]{625y^8z^4}$, $\sqrt[12]{\frac{1}{64}y^6z^{12}}$, $\sqrt[12]{\frac{1}{27}y^9}$.
11. $\sqrt[3]{16}$.
12. $\sqrt{5}$.
13. $\sqrt{3}$.
14. $\sqrt[4]{48}$.

Pages 99, 100.

1. $\sqrt{35}$.
2. 6.
3. $8\sqrt{2}$.
4. $3\sqrt[6]{2}$.
5. $7\sqrt[3]{3}$.
6. $2\sqrt[4]{5}$.
7. $\sqrt[4]{360}$.
8. $\sqrt{\frac{1}{6}}$.
9. $\sqrt{\frac{1}{2}}$.
10. 1.
11. $\frac{1}{6}$.
12. $\sqrt[6]{72}$.
13. $x\sqrt[20]{x^3}$.
14. $n\sqrt[6]{200n}$.
15. $t\sqrt[12]{1728t^4}$.
16. $2\sqrt[3]{3}$.
17. $y^2\sqrt[6]{360}$.
18. $V\sqrt[12]{648V^7}$.
19. $\sqrt{a^2-ab}$.
20. $\sqrt{m^2-n^2}$.
21. $(x+y)\sqrt{x-y}$.
22. 1.
23. $28+17\sqrt{6}$.
24. $-1-\sqrt{10}$.
25. $2+2\sqrt{3}$.
26. $2\sqrt{6}$.
27. $a+2b+3\sqrt{ab}$.
28. $m\sqrt{mn}-n\sqrt{mn}-m\sqrt{n}+n\sqrt{m}$.
29. $x+\sqrt{xy}+y$.
30. $n+14+8\sqrt{n-1}$.
31. $a+b$.
32. $p+q+r+2\sqrt{pq}+2\sqrt{pr}+2\sqrt{qr}$.
33. $\frac{1}{4}R^2\sqrt{2}$.
34. $\frac{1}{2}R\sqrt{2+\sqrt{2}}$.
35. $\sqrt{12}$.
36. $\sqrt{175}$.
37. $\sqrt{\frac{1}{2}}$.
38. $\sqrt[3]{16}$.
39. $\sqrt[3]{\frac{1}{9}}$.
40. $\sqrt{a^3b}$.
41. $\sqrt[4]{2t^{13}}$.
42. $\sqrt[3]{64m^3n^5}$.
43. $\sqrt{\frac{a}{b}}$.
44. $\sqrt[3]{\frac{2v}{t^5}}$.
45. $\sqrt[4]{\frac{c^2}{k}}$.

Pages 102, 103.

1. $\frac{2}{3}\sqrt{3}$.
2. $\frac{1}{5}\sqrt{10}$.
3. $\frac{4}{3}\sqrt{6}$.
4. $2\sqrt{2}$.
5. $\frac{3}{7}\sqrt{21}$.
6. $\frac{2}{5}\sqrt{5}$.
7. $\frac{1}{2}\sqrt{2}$.
8. $\frac{5}{2}\sqrt[3]{4}$.
9. $2\sqrt[3]{9}$.
10. $\frac{1}{2}\sqrt[3]{4}$.
11. $\frac{1}{n}\sqrt{mn}$.
12. $5\sqrt{2r}$.
13. $3\sqrt{5V}$.
14. $\frac{1}{2y}\sqrt[3]{16y^2}$.
15. $\sqrt[6]{x}$.
16. $18\sqrt{3}-18$.
17. $-\frac{5}{2}-\frac{5}{2}\sqrt{5}$.
18. $2-\sqrt{2}$.
19. $2+\sqrt{3}$.

20. $\frac{4}{5}+\frac{2}{5}\sqrt{10}-\frac{1}{5}\sqrt{6}-\frac{1}{5}\sqrt{15}$.

21. $\frac{14}{13}\sqrt{42}-\frac{87}{13}$.

22. $\left(\frac{1}{m-n}\right)(\sqrt{m}+\sqrt{n})$.

23. $\sqrt{a}-\sqrt{b}$.

24. $\sqrt{x^3}+\sqrt{y^3}$.

25. $\frac{1}{2}\sqrt{3}-\frac{1}{4}\sqrt{2}-\frac{1}{4}\sqrt{6}$.

26. $\frac{5}{59}\sqrt{7}-\frac{25}{59}\sqrt{5}+\frac{45}{59}\sqrt{3}+\frac{10}{59}\sqrt{105}$.

27. $\frac{1}{2}\sqrt{10}-\frac{1}{2}\sqrt{6}$.

28. $\dfrac{p^2-pq-pr-p\sqrt{pr}-q\sqrt{pr}+r\sqrt{pr}-p\sqrt{pq}+q\sqrt{pq}-r\sqrt{pq}+2p\sqrt{qr}}{p^2+q^2+r^2-2pq-2pr-2qr}$.

Page 104.

1. $3\sqrt{3}$.
2. 25.
3. $16\sqrt{2}$.
4. $16\sqrt[3]{4}$.
5. $405\sqrt[3]{5}$.
6. $x\sqrt[3]{x}$.
7. $N^2\sqrt[4]{N}$.
8. $t^7\sqrt[5]{t}$.
9. $8a^4b^4\sqrt{ab}$.
10. $36x^5$.
11. $10000m^{12}n^2\sqrt[5]{m^4n^2}$.
12. $243p^{17}q^{14}$.
13. $16(a^2-b^2)^2$.
14. $\sqrt[6]{5}$.
15. $\sqrt{2}$.
16. $\sqrt[3]{5}$.
17. 2.
18. $\sqrt[3]{2}$.
19. $\sqrt[4]{x}$.
20. $\sqrt[3]{5n^2}$.
21. $y\sqrt[4]{2x^2}$.
22. $7r^3s^4\sqrt{2}$.
23. $\sqrt[3]{a^2-b^2}$.
24. $\sqrt[p]{a}$.

Pages 105, 106.

1. $6i$.
2. $8i$.
3. $9i$.
4. $10i$.
5. $25i$.
6. $\sqrt{6}\sqrt{-1}$.
7. $2\sqrt{3}\sqrt{-1}$.
8. $2\sqrt{5}\sqrt{-1}$.
9. $\frac{1}{2}i$.
10. $\frac{1}{3}i$.
11. a^2bi.
12. $4mn^3i$.
13. $(a+b)i$.
14. $(3x-4y)i$.
15. $7i$.
16. $18i$.
17. $2i$.
18. $16i$.
19. $(\frac{5}{6}\sqrt{6}+1)i$.
20. $3a^2i$.
21. $3x+y\sqrt{-1}$.
22. $16+2i$.
23. $2m-5ni$.
24. $-4-23i$.
25. $(a-c)-(b+d)i$.

Pages 107, 108.

1. i.
2. 1.
3. $-i$.
4. i.
5. 1.
6. -1.
7. $-i$.
8. i.
9. -10.
10. -12.

11. -42.
12. -72.
13. -10.
14. $-ab$.
15. $-2\sqrt{15}$.
16. -30.
17. $-14i$.
18. $-18i$.
19. $-\sqrt{70}\sqrt{-1}$.
20. $-n^4\sqrt{n}\,\sqrt{-1}$.
21. $420i$.
22. -1080.
23. 3.
24. -13.
25. 3.
26. 1.
27. -169.
28. $b-a$.
29. $-x^3-y^3+2xy\sqrt{xy}$.
30. a^2+b^2.

Pages 108, 109.

1. $-5i$.
2. $-\frac{3}{2}i$.
3. $-5i$.
4. $-7i$.
5. $-\sqrt{n}\,\sqrt{-1}$.
6. 2.
7. $\frac{4}{3}$.
8. 2.
9. $\frac{1}{3}i$.
10. $\frac{5}{6}a$.
11. $\frac{5}{7}v$.
12. $\frac{3}{2x}$.
13. $\frac{1}{5}-\frac{3}{5}i$.
14. $\frac{1}{3}+\frac{1}{3}\sqrt{-3}+\frac{1}{3}\sqrt{-2}-\frac{1}{3}\sqrt{6}$.
15. $\frac{2}{7}-\frac{3}{7}\sqrt{-5}$.
16. $\frac{2}{9}-\frac{4}{9}\sqrt{3}+\frac{1}{9}\sqrt{-6}+\frac{4}{9}\sqrt{-2}$.
17. $\frac{1}{9}-\frac{2}{45}\sqrt{-5}$.
18. $\sqrt{10}+2\sqrt{15}+3\sqrt{6}+3$.
19. $\frac{x^2+2\sqrt{-y}-y}{x^2+y}$.
20. $\frac{a^2-b^2}{a^2+b^2}+\frac{2ab}{a^2+b^2}i$.

Pages 110, 111.

1. 0.632.
2. 0.654.
3. 0.674.
4. 0.606.
5. 0.912.
6. 0.547.
7. 0.806.
8. a^2b^3.
9. $x^3\sqrt{y}$.
10. A^2B.
11. $x^2\sqrt{x}$.
12. $R\sqrt[n]{1-3\sqrt{5}}$.
13. $\sqrt[n]{a^2}-2\sqrt[n]{ab}+\sqrt[n]{b^2}$.
14. Yes.
15. Yes.
16. Yes.
17. Yes.
19. $\sqrt{7}-\sqrt{3}$.
20. $\sqrt{5}+\sqrt{6}$.
21. $\sqrt{5}+3$.
22. $\sqrt{12}-1$.
23. $2+\sqrt{2}$.
24. $4-\sqrt{10}$.
25. $\sqrt{7}+\sqrt{11}$.
26. $\frac{6+3\sqrt{-3}}{7}$.
27. $\frac{\sqrt{3}-\sqrt{-1}}{2}$.
28. $\frac{1-4\sqrt{-3}}{7}$.
29. $\frac{3\sqrt{-3}+3\sqrt{2}}{-5}$.
30. $\frac{x^2-y^2+2xy\sqrt{-1}}{x^2+y^2}$.
31. $-\sqrt{-1}$
32. $\frac{7}{4}+\frac{1}{4}\sqrt{-5}$.
33. $114+7\sqrt{-21}$.

Page 112.

1. $2x^2-10=0$; pure.
2. $4n^2+10n-2=0$; complete.
3. $12t^2-7t+21=0$; complete.
4. $3V^2-10V+12=0$; complete.
5. $4a^2-7a+6=0$; complete.
6. $y^2+y+1=0$; complete.
7. $2r^2-5r-3=0$; complete.
8. $5P^2-3=0$; pure.
9. $2a^2+1=0$; pure.
10. $r^2-4=0$; pure.
11. $5T^2=0$; pure.

Pages 113, 114.

1. $n=1$ or 4.
2. $k=4$ or 8.
3. $P=2$ or -8.
4. $A=-1$ or $1\frac{1}{2}$.
5. $v=2$ or $-\frac{3}{4}$.
6. $t=\frac{1}{2}$ or $-3\frac{1}{2}$.
7. $k=-\frac{2}{5}$ or $3\frac{1}{2}$.
8. $N=-\frac{1}{4}$ or $1\frac{3}{5}$.
9. $x=5$ or -3.
10. $p=5$ or $-4\frac{1}{6}$.
11. $d=3$ or $-1\frac{2}{3}$.
12. $x=4$ or -1.
13. $n=5$ or $-5\frac{1}{4}$.
14. $R=6$ or $-1\frac{1}{5}$.
15. $y=4$ or $-\frac{1}{6}$.
16. $h=-1$ or -9.
17. $D=6$ or -1.
18. $a=\frac{1}{5}$ or $8\frac{1}{2}$.
19. $x=2$ or -2.
20. $g=4$ or 5.

Pages 114, 115.

1. $y=\pm 2$.
2. $F=\pm 5$.
3. $k=\pm 3$.
4. $A=\pm\frac{3}{4}$.
5. $p=\pm\frac{1}{3}\sqrt{21}$.
6. $T=\pm\frac{1}{4}\sqrt{-2}$.
7. $n=\pm\frac{1}{5}\sqrt{-30}$.
8. $G=\pm\frac{1}{2}\sqrt{-1}$.
9. $b=\pm\sqrt{3}$.
10. $R=\pm\frac{4}{5}\sqrt{-5}$.
11. $x=\pm 3$.
12. $N=\pm\frac{1}{2}\sqrt{58}$.
13. $a=\pm 4$.
14. $z=\pm 2\sqrt{5}$.
15. $A=\pm\sqrt{-23}$.
16. $n=\pm\frac{1}{2}\sqrt{13}$.
17. $t=\pm 1$.
18. $a=\pm 3\sqrt{-1}$.
19. $K=\pm 7$.
20. $m=\pm 6$.
21. $B=\pm\frac{1}{6}\sqrt{39}$.
22. $x=\pm 2.45$.
23. $M=\pm 1.32$.
24. $t=\pm 0.77$.
25. $t=\pm 0.45$.
26. $D=\pm 3.06$.
27. $s=\pm 1.41$.
28. $H=\pm 6.56$.

Pages 116, 117.

1. $n=2$ or -12.
2. $A=-3$ or $\frac{1}{4}$.
3. $x=-4$ or $1\frac{2}{3}$.
4. $v=8$ or $\frac{1}{6}$.
5. $T=-3$ or $\frac{5}{12}$.
6. $y=-2$ or $-\frac{2}{3}$.
7. $N=\frac{2}{3}$ or $-\frac{1}{2}$.
8. $k=1\frac{1}{2}$ or $-\frac{4}{5}$.
9. $D=\frac{6}{7}\pm\frac{1}{7}\sqrt{57}$.
10. $s=3$ or $-\frac{1}{4}$.
11. $z=2\frac{1}{2}$ or $\frac{2}{5}$.
12. $H=\frac{7}{12}\pm\frac{1}{12}\sqrt{13}$.
13. $t=1\frac{1}{2}$ or $\frac{2}{3}$.
14. $R=-1$ or $\frac{1}{2}$.
15. $V=7$ or $-1\frac{1}{3}$.
16. $m=\frac{1}{3}\pm\sqrt{3}$.
17. $G=1$ or $-\frac{5}{6}$.
18. $E=\frac{1}{3}\pm\frac{2}{3}\sqrt{-5}$.
19. $x=\frac{5}{4}\pm\frac{1}{4}\sqrt{3}$.
20. $n=2$ or $-1\frac{1}{3}$.
21. $v=-1\pm\frac{1}{2}\sqrt{-10}$.
22. $a=-15\pm\sqrt{298}$.
23. $m=-7$ or $-1\frac{1}{6}$.
24. $d=5$ or $1\frac{1}{2}$.
25. $y=1$ or $2\frac{2}{5}$.
26. $N=\frac{21}{5}\pm\frac{1}{5}\sqrt{21}$.
27. $t=4$ or $3\frac{2}{3}$.
28. $x=-2$ or $-1\frac{20}{29}$.
29. $a=-7.46$ or -0.54.
30. $V=-4.79$ or -0.21.
31. $n=1.28$ or -0.78.
32. $D=0.70$ or -1.20.

Pages 118, 119.

1. $Q=3$ or 5.
2. $z=9$ or -7.
3. $a=-4$ or -12.
4. $x=-1$ or $2\frac{1}{2}$.
5. $v=-6$ or $-\frac{1}{6}$.
6. $N=\frac{2}{3}$ or $-3\frac{1}{2}$.
7. $b=\frac{2}{3}$ or $-\frac{4}{5}$.
8. $y=1\frac{1}{3}$ or $\frac{5}{7}$.
9. $t=\frac{7}{2}\pm\frac{1}{2}\sqrt{41}$.
10. $B=-\frac{1}{16}\pm\frac{1}{16}\sqrt{161}$.
11. $n=3\pm\frac{1}{2}\sqrt{22}$.
12. $V=\frac{1}{40}\pm\frac{1}{40}\sqrt{-1039}$.
13. $P=2$ or $-1\frac{1}{3}$.
14. $k=\frac{1}{3}$ or -4.
15. $R=\frac{1}{18}\pm\frac{1}{18}\sqrt{217}$.
16. $x=\frac{3}{2}\pm\frac{1}{2}\sqrt{73}$.
17. $x=-8\pm8\sqrt{2}$.
18. $p=\frac{13}{6}\pm\frac{1}{6}\sqrt{13}$.
19. $n=-\frac{1}{2}\pm\frac{1}{2}\sqrt{-3}$.
20. $n=\frac{1}{2}\pm\frac{1}{2}\sqrt{-3}$.
21. $v=0$ or 2.
22. $y=2\pm\sqrt{5}$.
23. $g=\frac{3}{4}\pm\frac{1}{4}\sqrt{-7}$.
24. $R=3$ or $-\frac{1}{3}$.
25. $F=3$ or $-6\frac{1}{2}$.
26. $b=6$ or -1.
27. $H=-\frac{1}{2}\pm\frac{1}{2}\sqrt{5}$.
28. $n=\frac{5}{2}\pm\frac{1}{2}\sqrt{-7}$
29. $c=3$ or $\frac{2}{5}$.
30. $r=3$ or $\frac{3}{4}$.
31. $V=4$ or $-\frac{4}{7}$.
32. $n=3$ or -5.
33. $A=\frac{1}{2}$ or $-4\frac{2}{3}$.
34. $Q=4$ or -1.
35. $s=-13$ or $4\frac{1}{3}$.
36. $f=-5$ or -1.
37. $S=1.28$ or -0.78.
38. $v=0.86$ or -0.46.
39. $n=0.30$ or -1.13.
40. $x=5.16$ or -1.16.
41. $T=1.19$ or -1.69.

Pages 119, 120.

1. $v = \pm a$.

2. $x = 2n$ or $-n$; $n = \frac{1}{2}x$ or $-x$.

3. $x = \frac{1}{2}a \pm \frac{1}{2}\sqrt{a^2 - 4b}$.

4. $v = -s \pm \sqrt{s^2 - t}$.

5. $m = a$ or b.

6. $t = 2m$ or $-n$.

7. $y = b$ or $b + 2$.

8. $P = -1$ or $\dfrac{1}{n-1}$.

9. $n = 2a \pm a\sqrt{2}$.

10. $R = -2m$ or $\dfrac{m-n}{2n}$.

11. $t = \dfrac{ac - bd \pm \sqrt{(a^2 - b^2)(c^2 - d^2)}}{ad - bc}$.

12. $r = \dfrac{-(mn + mp + np) \pm \sqrt{m^2n^2 + m^2p^2 + n^2p^2 - m^2np - mn^2p - mnp^2}}{m + n + p}$.

13. $V = 0$ or $n^3 - n$.

14. $N = \dfrac{k^2 \pm \sqrt{k^4 + d^2f^2}}{f}$.

15. $E = \pm\sqrt{\dfrac{JWhR}{t}}$.

16. $C = \pm 5\sqrt{\dfrac{H}{6Rt}}$.

17. $L = \pm\dfrac{1}{2n}\sqrt{\dfrac{Eg}{w}}$.

18. $D = \pm d\sqrt{\dfrac{f}{F}}$.

19. $D = \dfrac{KN(R+2) \pm \sqrt{K^2N^2R^2 + 4K^2N^2R + 4K^2N^2 + 4KNRH + 8KNH}}{2KN(R+2)}$.

20. $t = \dfrac{-513 \pm \sqrt{263169 + 4H}}{2}$.

21. $r = \pm\sqrt{\dfrac{\pi r'^2 + A}{\pi}}$; $r' = \pm\sqrt{\dfrac{\pi r^2 - A}{\pi}}$.

22. $R = \dfrac{-Hr \pm \sqrt{12HV - 3H^2r^2}}{2H}$; $r = \dfrac{-HR \pm \sqrt{12HV - 3H^2R^2}}{2H}$.

Pages 121, 124.

1. $12\frac{1}{3}$; $17\frac{2}{3}$.

2. 10 in.; 8 in.

3. 16 in.; 12 in.

4. 20 in.; 24 in.

5. 768 sq. in.

6. 20 in.

7. 20 in. by 40 in.

8. 6 in.; 4 in.

9. 30 rd.; 60 rd.

10. 24 ft.; 18 ft.

11. $PC = 12.2-$.

12. 3 mi. 4712 ft.

13. 5.414 in.; 2.586 in.

14. 0.546 ft.

15. 7.64 rd.

16. $45\frac{1}{2}$ ft.

17. 12 in.

18. 1200 sq. rd.

19. 1500 bu.

20. 140 acres.

21. 14.

22. 100 ft.

23. 35 ft.

24. 6 in.; 8 in.

25. 24 min.; 18 min.
26. $15\frac{5}{9}$ min.
27. 45 mi. per hour; 30 mi. per hour.
28. $2\frac{1}{2}$ hr.
29. 12.
30. 1 mi. per hour; 2 mi. per hour.
31. 64 sq. in.
32. 4 in.
33. 14 in.
34. 582.74 – sq. rd.
35. 1 ft.

Page 125.

1. Real, unequal, and free of surds.
2. Real, unequal, and involve surds.
3. Imaginary.
4. Real, unequal, and free of surds.
5. Imaginary.
6. Imaginary.
7. Imaginary.
8. Real, unequal, and free of surds.
9. Real and equal.
10. Imaginary.
11. Real, unequal, and involve surds.
12. Real and equal.
13. Real, unequal, and involve surds.
14. Real, unequal, and involve surds.
16. $c = -2\frac{1}{4}$.
17. $c = \pm 20$.
18. $c = \pm 16$.
19. $c = \frac{1}{3}$.
20. $c = 8$.
21. $c = 5\frac{1}{3}$.
22. $4mn = 25$.
23. $n^2 = 16m$.
24. $m^2 = 4n$.
25. $m^2 = 8n$.

Pages 126, 127.

1. $x^2 - 8x + 12 = 0$.
2. $x^2 - 10x + 9 = 0$.
3. $x^2 + 8x + 15 = 0$.
4. $x^2 - 6x - 16 = 0$.
5. $x^2 + x - 42 = 0$.
6. $x^2 + 14x + 24 = 0$.
7. $x^2 - 5x - 50 = 0$.
8. $x^2 + 23x + 60 = 0$.
9. $x^2 + 2x - 99 = 0$.
10. $6x^2 - 5x + 1 = 0$.
11. $25x^2 - 15x - 4 = 0$.
12. $24x^2 - x - 3 = 0$.
13. $24x^2 + 26x + 5 = 0$.
14. $x^2 - 4ax + 3a^2 = 0$.
15. $x^2 + 2nx - 5mx - 10mn = 0$.
16. $x^2 - 2x - 4 = 0$.
17. $x^2 + 8x + 13 = 0$.
18. $x^2 - 2x + 7 = 0$.
19. $x^2 + 6x + 13 = 0$.
20. $n = 2$ or -3.
21. $t = \frac{5}{2} \pm \frac{1}{2}\sqrt{-3}$.
22. $a = 1\frac{2}{3}$ or -1.
23. $V = \frac{1}{10} \pm \frac{1}{10}\sqrt{-19}$.
24. $N = 4$ or -1.
25. $r = \frac{5}{4} \pm \frac{5}{4}\sqrt{-7}$.
26. $A = -\frac{1}{2} \pm \frac{1}{2}\sqrt{37}$.
27. $x = \frac{1}{2}a \pm \frac{1}{2}\sqrt{a^2 - 12b}$.

Pages 128, 129.

1. $x=5$ or 7.

2. $x=8$ or -9.

3. $x=-6$ or -8.

4. $x=8$ or $3\frac{1}{2}$.

5. $x=2$ or $-6\frac{2}{3}$.

6. $x=-5$ or $4\frac{2}{3}$.

7. Real, unequal, and free of surds. Graph meets x-axis at two distinct points.

8. Same as Ex. 7.

9. Real and equal. Graph touches x-axis at one point.

10. Imaginary. Graph does not meet x-axis.

11. Same as Ex. 10.

12. Same as Ex. 9.

Pages 129, 130.

1. Real and free of surds.

2. Real and free of surds.

3. Imaginary.

4. Imaginary.

5. Real and free of surds.

6. Real, involving surds.

7. Imaginary.

8. Real, involving surds.

9. Imaginary.

10. Real and free of surds.

12. $2(S-\frac{1}{4}-\frac{1}{4}\sqrt{-15})(S-\frac{1}{4}+\frac{1}{4}\sqrt{-15})$.

13. $5(P+\frac{4}{5}-\frac{2}{5}\sqrt{-1})(P+\frac{4}{5}+\frac{2}{5}\sqrt{-1})$.

14. $3(b-\frac{1}{6}-\frac{1}{6}\sqrt{13})(b-\frac{1}{6}+\frac{1}{6}\sqrt{13})$.

15. $9(d+\frac{2}{9}-\frac{1}{9}\sqrt{-5})(d+\frac{2}{9}+\frac{1}{9}\sqrt{-5})$.

16. $(n-11)(n+13)$.

17. $(2y-1)(5y+8)$.

18. $7(A-\frac{1}{14}-\frac{1}{14}\sqrt{-447})(A-\frac{1}{14}+\frac{1}{14}\sqrt{-447})$.

19. $(T+1-\sqrt{-14})(T+1+\sqrt{-14})$.

Pages 132, 133.

1. $n=1,\ -\frac{1}{2}\pm\frac{1}{2}\sqrt{-3}$.

2. $a=1,\ -1,\ \frac{1}{2}\pm\frac{1}{2}\sqrt{-3},$
$-\frac{1}{2}\pm\frac{1}{2}\sqrt{-3}$.

3. $V=3,\ -\frac{3}{2}\pm\frac{3}{2}\sqrt{-3}$.

4. $x=1,\ -1,\ \pm\sqrt{-1}$.

5. $y=3,\ -3,\ \pm 3\sqrt{-1}$.

6. $t=1,\ -1,\ 2,\ -2$.

7. $k=2,\ -2,\ \pm 2\sqrt{-1}$.

8. $N=\pm\sqrt{5},\ \pm\sqrt{-3}$.

9. $s=1\frac{1}{2},\ -\frac{3}{4}\pm\frac{3}{4}\sqrt{-3}$.

10. $A=-\frac{5}{4},\ \frac{5}{8}\pm\frac{5}{8}\sqrt{-3}$.

11. $g=2,\ -2,\ \pm\frac{1}{2}\sqrt{-6}$.

12. $m=1,\ -2,\ 1\pm\sqrt{-3},$
$-\frac{1}{2}\pm\frac{1}{2}\sqrt{-3}$.

13. $E=\pm\frac{1}{2}\sqrt{6},\ \pm\frac{1}{2}\sqrt{14}$.

14. $c=\pm\frac{2}{3}\sqrt{-3},\ \pm\frac{1}{3}\sqrt{-6}$.

15. $x=1,\ -1,\ -2$.

16. $z=1,\ 2,\ -2$.

17. $T=3,\ \pm\sqrt{-1}$.

18. $a=-5,\ +2,\ -2$.

19. $R=-4,\ -1,\ \frac{1}{2}\pm\frac{1}{2}\sqrt{-3}.$
20. $h=1,\ \pm\sqrt{-1},\ -\frac{1}{2}\pm\frac{1}{2}\sqrt{-3}.$
21. $P=0,\ -2,\ \frac{5}{6}\pm\frac{1}{6}\sqrt{-119}.$
22. $n=0,\ 1,\ 1,\ 1.$
23. $4,\ -2,\ \pm2\sqrt{-3}.$
24. $3,\ -3,\ \pm3\sqrt{-1}.$
25. $1,\ -1,\ \frac{1}{2}\pm\frac{1}{2}\sqrt{-3},\ -\frac{1}{2}\pm\frac{1}{2}\sqrt{-3}.$
26. $2,\ -2,\ 1\pm\sqrt{-3},\ -1\pm\sqrt{-3}.$

Pages 134, 135.

1. $x=\pm\sqrt{6},\ \pm2\sqrt{-2}.$
2. $a=\pm\frac{1}{2}\sqrt{6},\ \pm\frac{1}{3}\sqrt{-15}.$
3. $V=\pm\frac{1}{3}\sqrt{6},\ \pm\frac{1}{4}\sqrt{-10}.$
4. $n=-1,\ -2,\ 1\pm\sqrt{-3},\ \frac{1}{2}\pm\frac{1}{2}\sqrt{-3}.$
5. $p=2,\ -1\frac{1}{2},\ -1\pm\sqrt{-3},$
$\frac{3}{4}\pm\frac{3}{4}\sqrt{-3}.$
6. $y=1,\ 1,\ -1,\ -1,\ \pm\sqrt{-1},$
$\pm\sqrt{-1}.$
7. $x=2,\ -2,\ 3,\ -3.$
8. $t=-1,\ -1,\ 1,\ -3.$
9. $n=-1,\ 2,\ -2,\ 3.$
10. $P=-\frac{7}{4}\pm\frac{1}{4}\sqrt{89},\ -\frac{7}{4}\pm\frac{1}{4}\sqrt{33}.$
11. $x=-\frac{1}{2}\pm\frac{1}{2}\sqrt{5},\ -\frac{1}{4}\pm\frac{1}{4}\sqrt{17}.$
12. $m=\frac{1}{4}\pm\frac{1}{4}\sqrt{-7},\ \frac{1}{4}\pm\frac{1}{4}\sqrt{-7}.$
13. $d=1\pm\sqrt{2},\ 1\pm\sqrt{2}.$
14. $A=\pm1,\ \pm\sqrt{-3}.$
15. $c=1,\ -2,\ -\frac{1}{2}\pm\frac{1}{2}\sqrt{-3}.$
16. $H=0,\ 0,\ -3,\ -3.$
17. $n=\frac{5}{2}\pm\frac{1}{2}\sqrt{17},\ \frac{5}{2}\pm\frac{1}{2}\sqrt{23}.$
18. $v=-\frac{3}{2}\pm\frac{1}{2}\sqrt{5},\ \frac{3}{2}\pm\sqrt{3}.$
19. $R=\frac{1}{2}\pm\frac{3}{2}\sqrt{-3},\ \frac{1}{2}\pm\frac{1}{2}\sqrt{-7}.$
20. $b=\frac{3}{2}\pm\frac{1}{2}\sqrt{-11},\ \frac{3}{2}\pm\frac{1}{2}\sqrt{17}.$
21. $W=0,\ 0,\ -5,\ -5.$
22. $x=2,\ 4,\ -\frac{7}{2}\pm\frac{1}{2}\sqrt{17}.$
23. $n=3\pm\sqrt{14},\ -\frac{5}{2}\pm\frac{3}{2}\sqrt{5}.$
24. $a=1,\ 1,\ -\frac{3}{2}\pm\frac{1}{2}\sqrt{5}.$
25. $y=1,\ -1,\ -\frac{3}{2}\pm\frac{1}{2}\sqrt{13}.$
26. $D=1,\ 1,\ \frac{1}{2}\pm\frac{1}{2}\sqrt{-3}.$

Page 136.

2. $x=3,\ 5,\ 7.$
3. $x=-1,\ 6,\ -5.$
4. $x=2,\ -2,\ 5,\ -5.$
5. $x=-2,\ 4,\ 8,\ -8.$
6. One real; two imaginary.
7. All real; two equal.

Pages 140, 141.

1. $x=3.$
2. $y=5.$
3. $t=61.$
4. $x=14.$
5. $n=9.$
6. $a=-5.$
7. $v=14.$
8. $m=4.$
9. $R=7.$

10. $k=7$.
11. $D=-2$.
12. $x=-8, 40$.
13. $M=-4$.
14. $t=5, \frac{5}{49}$.
15. $A=144$.
16. $d=16$.
17. $s=9$.
18. $n=-2\frac{2}{3}$.
19. $n=2, 3$.
20. Impossible.
21. $h=4$.
22. Impossible.
23. Impossible.
24. Impossible.
25. $a=1$.
26. Impossible.
27. $p=8$.
28. $a=-\frac{10}{13}$.
29. $x=\frac{13}{6}\pm\frac{1}{6}\sqrt{13}$.
30. $t=0, -1$.
31. $y=0, \pm\sqrt{-1}$.
32. $x=4$.
33. $x=1\frac{1}{3}$.
34. $n=6\frac{3}{4}$.
35. $a=\frac{1}{3}$.
36. $s=9$.
37. $k=1$.
38. $R=1$.
39. $n=\frac{4}{5}$.
40. $a=16$.
41. $A=0, -12$.
42. $N=2$.
43. $r=3$.
44. $x=3, -1\frac{2}{3}$.
45. $n=0, -3$.
46. $t=3, -\frac{1}{2}$.
47. $r=5, -4\frac{2}{3}$.
48. $a=4, 20$.
49. Impossible.
50. $l=\frac{t^2g}{\pi^2}$.
51. $s=\frac{v^2}{2g}$.
52. $t=\frac{n^2T}{N^2}$; $T=\frac{N^2t}{n^2}$.
53. $s=ut-\frac{1}{2}gt^2$.

Page 142.

1. $x=1$. **2.** $x=5$. **3.** $x=6$. **4.** $x=4$.

Pages 144, 145.

1. $r=2, s=1$; $r=-1, s=-2$.
2. $h=4, k=2$; $h=2, k=4$.
3. $a=5, b=1$; $a=-7, b=-5$.
4. $x=1, y=3$; $x=2\frac{1}{2}, y=-3$.
5. $m=7, n=2$; $m=2, n=7$.
6. $x=4, y=2$; $x=3\frac{1}{7}, y=-\frac{4}{7}$.
7. $r=10, R=-1$; $r=-4, R=2\frac{1}{2}$.
8. $t=-\frac{1}{2}\pm\frac{1}{6}\sqrt{-951}$,
$v=\frac{3}{4}\pm\frac{1}{4}\sqrt{-951}$.
9. $N=2, P=\frac{1}{2}$; $N=\frac{1}{2}, P=2$.
10. $a=5, b=-5$; $a=0, b=-10$.
11. $T=15, V=10$;
$T=-16\frac{1}{3}, V=-10\frac{7}{9}$.
12. $x=-4, y=-3$; $x=\frac{2}{3}, y=-1\frac{4}{9}$.
13. $m=2+8\sqrt{-1}, n=3-8\sqrt{-1}$;
$m=2-8\sqrt{-1}, n=3+8\sqrt{-1}$.
14. $k=7, n=4$; $k=-4, n=-7$.
15. $x=2, y=2$; $x=2, y=2$.
16. $A=1\frac{1}{3}, B=\frac{2}{3}$; $A=\frac{1}{2}, B=1\frac{1}{2}$.
17. $p=8, q=-11$; $p=11, q=-8$.
18. $t=4, u=12$; $t=-5\frac{1}{7}, u=-1\frac{5}{7}$.
19. $P=4, Q=5$; $P=5, Q=4$.

20. $x=\frac{1}{2}a+\frac{1}{2}\sqrt{5a^2+4b^2}$,
$y=-\frac{1}{2}a+\frac{1}{2}\sqrt{5a^2+4b^2}$;
$x=\frac{1}{2}a-\frac{1}{2}\sqrt{5a^2+4b^2}$,
$y=-\frac{1}{2}a-\frac{1}{2}\sqrt{5a^2+4b^2}$.

21. $x=\frac{1}{2}m+\frac{1}{2}n$, $y=\frac{1}{2}m-\frac{1}{2}n$;
$x=\frac{1}{2}m-\frac{1}{2}n$, $y=\frac{1}{2}m+\frac{1}{2}n$.

22. $x=\frac{1}{2}h+\frac{1}{2}\sqrt{h^2+4l^2}$,
$y=-\frac{1}{2}h+\frac{1}{2}\sqrt{h^2+4l^2}$;
$x=\frac{1}{2}h-\frac{1}{2}\sqrt{h^2+4l^2}$,
$y=-\frac{1}{2}h-\frac{1}{2}\sqrt{h^2+4l^2}$.

23. $x=\frac{1}{2}a+\frac{1}{2}\sqrt{2b-a^2}$,
$y=\frac{1}{2}a-\frac{1}{2}\sqrt{2b-a^2}$;
$x=\frac{1}{2}a-\frac{1}{2}\sqrt{2b-a^2}$,
$y=\frac{1}{2}a+\frac{1}{2}\sqrt{2b-a^2}$.

24. $c^2=4\pi S$.

Pages 146, 147.

1. $x=4, k=1$; $x=4, k=-1$; $x=-4, k=1$; $x=-4, k=-1$.

2. $A=\sqrt{2}$, $B=\sqrt{3}$; $A=\sqrt{2}$, $B=-\sqrt{3}$; $A=-\sqrt{2}$, $B=\sqrt{3}$; $A=-\sqrt{2}$, $B=-\sqrt{3}$.

3. $r=4$, $s=3$; $r=4$, $s=-3$; $r=-4$, $s=3$; $r=-4$, $s=-3$.

4. $m=\frac{8}{5}\sqrt{105}$, $n=\frac{8}{5}\sqrt{-5}$; $m=\frac{3}{5}\sqrt{105}$, $n=-\frac{8}{5}\sqrt{-5}$; $m=-\frac{3}{5}\sqrt{105}$, $n=\frac{8}{5}\sqrt{-5}$; $m=-\frac{3}{5}\sqrt{105}$, $n=-\frac{8}{5}\sqrt{-5}$.

5. $W=\frac{6}{13}\sqrt{65}$, $V=\frac{2}{13}\sqrt{91}$; $W=\frac{6}{13}\sqrt{65}$, $V=-\frac{2}{13}\sqrt{91}$; $W=-\frac{6}{13}\sqrt{65}$, $V=\frac{2}{13}\sqrt{91}$; $W=-\frac{6}{13}\sqrt{65}$, $V=-\frac{2}{13}\sqrt{91}$.

6. $x=5$, $y=3$; $x=5$, $y=-3$; $x=-5$, $y=3$; $x=-5$, $y=-3$.

7. $v=6$, $t=1$; $v=6$, $t=-1$; $v=-6$, $t=1$; $v=-6$, $t=-1$.

8. $R=5$, $D=2$; $R=5$, $D=-2$; $R=-5$, $D=2$; $R=-5$, $D=-2$.

9. $a=\frac{1}{2}\sqrt{2}$, $b=\frac{1}{3}\sqrt{2}$; $a=\frac{1}{2}\sqrt{2}$, $b=-\frac{1}{3}\sqrt{2}$; $a=-\frac{1}{2}\sqrt{2}$, $b=\frac{1}{3}\sqrt{2}$; $a=-\frac{1}{2}\sqrt{2}$, $b=-\frac{1}{3}\sqrt{2}$.

10. $P=2$, $Q=3$; $P=2$, $Q=-3$; $P=-2$, $Q=3$; $P=-2$, $Q=-3$.

11. $x=\frac{2}{3}$, $y=5$; $x=\frac{2}{3}$, $y=-5$; $x=-\frac{2}{3}$, $y=5$; $x=-\frac{2}{3}$, $y=-5$.

12. $E=6$, $R=\frac{4}{5}$; $E=6$, $R=-\frac{4}{5}$; $E=-6$, $R=\frac{4}{5}$; $E=-6$, $R=-\frac{4}{5}$.

13. $C=6$, $y=9$; $C=6$, $y=-9$; $C=-6$, $y=9$; $C=-6$, $y=-9$.

14. $r=4$, $r'=2$; $r=4$, $r'=-2$; $r=-4$, $r'=2$; $r=-4$, $r'=-2$.

15. $x=\frac{1}{2}$, $y=\frac{1}{4}$; $x=\frac{1}{2}$, $y=-\frac{1}{4}$; $x=-\frac{1}{2}$, $y=\frac{1}{4}$; $x=-\frac{1}{2}$, $y=-\frac{1}{4}$.

Pages 148, 149.

1. $x=2$, $y=1$; $x=-2$, $y=-1$; $x=\frac{3}{2}\sqrt{2}$, $y=\frac{1}{2}\sqrt{2}$; $x=-\frac{3}{2}\sqrt{2}$, $y=-\frac{1}{2}\sqrt{2}$.
2. $a=6$, $b=3$; $a=-6$, $b=-3$; $a=\frac{9}{4}\sqrt{6}$, $b=\frac{3}{4}\sqrt{6}$; $a=-\frac{9}{4}\sqrt{6}$, $b=-\frac{3}{4}\sqrt{6}$.
3. $m=2$, $n=-6$; $m=-1\frac{7}{9}$, $n=5\frac{1}{3}$; $m=-1+\sqrt{33}$, $n=-1+\sqrt{33}$; $m=-1-\sqrt{33}$, $n=-1-\sqrt{33}$.
4. $v=-5+\sqrt{91}$, $t=\frac{5}{2}-\frac{1}{2}\sqrt{91}$; $v=-5-\sqrt{91}$, $t=\frac{5}{2}+\frac{1}{2}\sqrt{91}$; $v=-\frac{19}{8}+\frac{1}{8}\sqrt{3001}$, $t=\frac{19}{40}-\frac{1}{40}\sqrt{3001}$; $v=-\frac{19}{8}-\frac{1}{8}\sqrt{3001}$, $t=\frac{19}{40}+\frac{1}{40}\sqrt{3001}$.
5. $x=4$, $y=4$; $x=-4$, $y=-4$; $x=4\sqrt{3}$, $y=-4\sqrt{3}$; $x=-4\sqrt{3}$, $y=4\sqrt{3}$.
6. $V=0$, $g=\sqrt{19}$; $V=0$, $g=-\sqrt{19}$; $V=3$, $g=-2$; $V=-3$, $g=2$.
7. $P=\sqrt{5}$, $Q=0$; $P=-\sqrt{5}$, $Q=0$; $P=5$, $Q=-3$; $P=-5$, $Q=3$.
8. $a=0$, $b=3\sqrt{3}$; $a=0$, $b=-3\sqrt{3}$; $a=3$, $b=-3$; $a=-3$, $b=3$.

Page 150.

1. $x=3$, $y=4$; $x=-3$, $y=-4$; $x=\frac{7}{3}\sqrt{3}$, $y=\frac{2}{3}\sqrt{3}$; $x=-\frac{7}{3}\sqrt{3}$, $y=-\frac{2}{3}\sqrt{3}$.
2. $m=2$, $n=4$; $m=-2$, $n=-4$; $m=\sqrt{2}$, $n=3\sqrt{2}$; $m=-\sqrt{2}$, $n=-3\sqrt{2}$.
3. $A=1$, $B=2$; $A=-1$, $B=-2$; $A=\sqrt{3}$, $B=0$; $A=-\sqrt{3}$, $B=0$.
4. $h=3$, $k=1$; $h=-3$, $k=-1$; $h=\frac{4}{5}\sqrt{10}$, $k=-\frac{3}{5}\sqrt{10}$; $h=-\frac{4}{5}\sqrt{10}$, $k=\frac{3}{5}\sqrt{10}$.
5. $v=2$, $w=-2$; $v=-2$, $w=2$; $v=\frac{4}{7}\sqrt{7}$, $w=-\frac{6}{7}\sqrt{7}$; $v=-\frac{4}{7}\sqrt{7}$, $w=\frac{6}{7}\sqrt{7}$.
6. $R=5$, $t=-2$; $R=-5$, $t=2$; $R=\frac{17}{5}\sqrt{-5}$, $t=\frac{7}{5}\sqrt{-5}$; $R=-\frac{17}{5}\sqrt{-5}$, $t=-\frac{7}{5}\sqrt{-5}$.
7. $x=1$, $y=5$; $x=-1$, $y=-5$; $x=14$, $y=-8$; $x=-14$, $y=8$.
8. $M=2$, $N=5$; $M=-2$, $N=-5$; $M=4\sqrt{3}$, $N=-\sqrt{3}$; $M=-4\sqrt{3}$, $N=\sqrt{3}$.
9. $a=2$, $b=5$; $a=-2$, $b=-5$; $a=4\sqrt{2}$, $b=3\sqrt{2}$; $a=-4\sqrt{2}$, $b=-3\sqrt{2}$.
10. $p=8$, $q=-5$; $p=-3$, $q=-5$; $p=3$, $q=5$; $p=-8$, $q=5$.
11. $F=5$, $C=4$; $F=-5$, $C=-4$; $F=\sqrt{3}$, $C=3\sqrt{3}$; $F=-\sqrt{3}$, $C=-3\sqrt{3}$.
12. $v=1$, $g=5$; $v=-1$, $g=-5$; $v=\frac{8}{5}\sqrt{-10}$, $g=-\frac{8}{5}\sqrt{-10}$; $v=-\frac{8}{5}\sqrt{-10}$, $g=\frac{8}{5}\sqrt{-10}$.

Page 152.

1. $x=-1, y=2$; $x=2, y=-1$. 2. $p=-2, q=5$; $p=5, q=-2$.

3. $a=6, b=1$; $a=-6, b=-1$; $a=1, b=6$; $a=-1, b=-6$.

4. $m=1, n=6$; $m=6, n=1$.

5. $P=3, Q=2$; $P=2, Q=3$; $P=-2, Q=-3$; $P=-3, Q=-2$.

6. $w=6, v=2$; $w=-2, v=-6$. 7. $x=5, y=-3$; $x=3, y=-5$.

8. $x=5, y=-1$; $x=-5, y=1$; $x=1, y=-5$; $x=-1, y=5$.

9. $A=1, B=-3$; $A=-1, B=3$; $A=3, B=-1$; $A=-3, B=1$.

10. $m=3, n=2$; $m=-2, n=-3$. 11. $s=3, t=1$; $s=1, t=3$.

12. $x=\frac{1}{2}\sqrt{5}+\frac{1}{2}, y=\frac{1}{2}\sqrt{5}-\frac{1}{2}$; $x=\frac{1}{2}\sqrt{5}-\frac{1}{2}, y=\frac{1}{2}\sqrt{5}+\frac{1}{2}$; $x=-\frac{1}{2}\sqrt{5}+\frac{1}{2}$,
$y=-\frac{1}{2}\sqrt{5}-\frac{1}{2}$; $x=-\frac{1}{2}\sqrt{5}-\frac{1}{2}, y=-\frac{1}{2}\sqrt{5}+\frac{1}{2}$.

13. $a=4, b=5$; $a=5, b=4$; $a=\sqrt{-14}-5, b=-\sqrt{-14}-5$;
$a=-\sqrt{-14}-5, b=\sqrt{-14}-5$.

14. $W=6, V=9$; $W=-9, V=-6$.

15. $r=1, r'=5$; $r=-1, r'=-5$; $r=-5, r'=-1$; $r=5, r'=1$.

16. $x=5, y=3$; $x=-5, y=-3$; $x=3, y=5$; $x=-3, y=-5$.

Page 154.

1. $m=2, n=3$; $m=3, n=2$.

2. $v=3, t=0$; $v=0, t=-3$.

3. $a=5, b=-2$; $a=2, b=-5$.

4. $x=4, y=1$; $x=1, y=4$.

5. $P=1, Q=3$; $P=3, Q=1$.

6. $a=1, b=3$; $a=3, b=1$.

7. $w=4, v=2$; $w=-2, v=-4$.

8. $M=3, N=2$; $M=-2, N=-3$.

9. $x=3, y=3$; $x=-3, y=-3$; $x=3, y=3$; $x=-3, y=-3$.

10. $r=3, s=1$; $r=-3, s=-1$; $r=1, s=3$; $r=-1, s=-3$.

11. $a=2, b=3$; $a=3, b=2$.

12. $m=0, n=0$; $m=\frac{1}{3}\sqrt{6}, n=-\frac{2}{3}\sqrt{6}$; $m=-\frac{1}{3}\sqrt{6}, n=\frac{2}{3}\sqrt{6}$.

13. $v=0, t=1$; $v=1, t=0$; $v=-2, t=3$.

14. $x=-7, y=7$; $x=3, y=2$; $x=1, y=3$.

15. $m=0, n=0$; $m=5, n=1$; $m=-5, n=-1$.

16. $a=2, b=7$; $a=\frac{1}{2}, b=1\frac{3}{4}$; $a=-\frac{1}{2}+\frac{1}{2}\sqrt{-3}, b=0$; $a=-\frac{1}{2}-\frac{1}{2}\sqrt{-3}, b=0$.

Pages 155, 156.

1. $m=1$, $n=2$; $m=2$, $n=1$; $m=\frac{3}{2}+\frac{1}{2}\sqrt{-55}$, $n=\frac{3}{2}-\frac{1}{2}\sqrt{-55}$; $m=\frac{3}{2}-\frac{1}{2}\sqrt{-55}$, $n=\frac{3}{2}+\frac{1}{2}\sqrt{-55}$.
2. $w=4$, $v=1$; $w=-1$, $v=-4$; $w=\frac{3}{2}+\frac{1}{2}\sqrt{-79}$, $v=-\frac{3}{2}+\frac{1}{2}\sqrt{-79}$; $w=\frac{3}{2}-\frac{1}{2}\sqrt{-79}$, $v=-\frac{3}{2}-\frac{1}{2}\sqrt{-79}$.
3. $A=1$, $B=3$; $A=3$, $B=1$; $A=2+5\sqrt{-1}$, $B=2-5\sqrt{-1}$; $A=2-5\sqrt{-1}$, $B=2+5\sqrt{-1}$.
4. $x=\frac{5}{2}+\frac{1}{2}\sqrt{-75+4\sqrt{353}}$, $y=\frac{5}{2}-\frac{1}{2}\sqrt{-75+4\sqrt{353}}$;
   $x=\frac{5}{2}+\frac{1}{2}\sqrt{-75-4\sqrt{353}}$, $y=\frac{5}{2}-\frac{1}{2}\sqrt{-75-4\sqrt{353}}$;
   $x=\frac{5}{2}-\frac{1}{2}\sqrt{-75+4\sqrt{353}}$, $y=\frac{5}{2}+\frac{1}{2}\sqrt{-75+4\sqrt{353}}$.
   $x=\frac{5}{2}-\frac{1}{2}\sqrt{-75-4\sqrt{353}}$, $y=\frac{5}{2}+\frac{1}{2}\sqrt{-75-4\sqrt{353}}$.
5. $n=3$, $m=2$; $n=2$, $m=3$; $n=6$, $m=-1$; $n=-1$, $m=6$;
   $n=\frac{5}{2}+\frac{1}{2}\sqrt{21}$, $m=\frac{5}{2}-\frac{1}{2}\sqrt{21}$; $n=\frac{5}{2}-\frac{1}{2}\sqrt{21}$, $m=\frac{5}{2}+\frac{1}{2}\sqrt{21}$;
   $n=\frac{5}{2}+\frac{1}{2}\sqrt{29}$, $m=\frac{5}{2}-\frac{1}{2}\sqrt{29}$; $n=\frac{5}{2}-\frac{1}{2}\sqrt{29}$, $m=\frac{5}{2}+\frac{1}{2}\sqrt{29}$.
6. $a=1$, $b=1$; $a=-1$, $b=-1$; $a=\frac{1}{2}\sqrt{-3}$, $b=\frac{1}{2}\sqrt{-3}$; $a=-\frac{1}{2}\sqrt{-3}$, $b=-\frac{1}{2}\sqrt{-3}$.
7. $h=2$, $k=2$; $h=-6$, $k=6$; $h=2\sqrt{-3}$, $k=3-\sqrt{-3}$; $h=-2\sqrt{-3}$, $k=3+\sqrt{-3}$; $h=\frac{12}{7}-\frac{6}{7}\sqrt{-3}$, $k=\frac{15}{7}+\frac{3}{7}\sqrt{-3}$; $h=\frac{12}{7}+\frac{6}{7}\sqrt{-3}$, $k=\frac{15}{7}-\frac{3}{7}\sqrt{-3}$.
8. $x=3$, $y=3$; $x=-3$, $y=-3$; $x=3\sqrt{-3}$, $y=-3\sqrt{-3}$; $x=-3\sqrt{-3}$, $y=3\sqrt{-3}$; and eight other complicated imaginary solutions.
9. $p=0$, $q=0$.

Pages 161, 162.

1. $x=6$, $y=3$; $x=-1\frac{1}{5}$, $y=6\frac{4}{5}$. 2. $x=9.1$, $y=4.1$; $x=-4.1$, $y=-9.1$.
3. $x=5.7$, $y=4.7$; $x=5.7$, $y=-4.7$; $x=-7.9$, $y=2.8$; $x=-7.9$, $y=-2.8$.
4. $x=4.5$, $y=4.1$; $x=4.5$, $y=-4.1$; $x=-3.5$, $y=2.9$; $x=-3.5$, $y=-2.9$.
5. $x=5.4$, $y=5.4$; $x=5.4$, $y=-5.4$; $x=-5.4$, $y=5.4$; $x=-5.4$, $y=-5.4$.
6. $x=6.96$, $y=5.70$; $x=6.96$, $y=-5.70$; $x=-6.96$, $y=5.70$; $x=-6.96$, $y=-5.70$.

7. $x=4.5, y=4$; $x=-4.5, y=4$; $x=3.6, y=-3$; $x=-3.6, y=-3$.

8. $x=7.5, y=6.6$; $x=7.5, y=-6.6$; $x=-7.5, y=6.6$; $x=-7.5, y=-6.6$.

Pages 162, 165.

1. 4; 9.
2. 12.8+ ft.; 7.8+ ft.
3. 50 rd.; 40 rd.
4. 30 ft.; 40 ft.
5. 20 yd.; 30 yd.
6. 18 ft.; 24 ft.
7. 12 in.; 16 in.; 20 in.
8. 16 ft.; 20 ft.
9. 6 in. by 6 in., or 9 in. by 4 in.
10. 2 in. and 1 in., or 3 in. and 2 in.
11. 6; 8.
12. \$96.
13. 40 ft.; 50 ft.
14. 25 ft.; 40 ft.
15. 10 da.; \$3.25.
16. 6 men; 8 da.
17. Apples \$32, potatoes \$22; or apples \$31.05, potatoes \$22.95.
18. \$250; 6%.
19. 8.4−.
20. $9\frac{3}{5}$; 96.
21. 24, \$200; 25, \$192.
22. 130 mi.; 117 mi.
23. A, $2\frac{1}{2}$ mi. per hour; B, 3 mi. per hour.
24. 48 rd.; 32 rd.
25. 20; 40.
26. 4 yd.; 5 yd.

Pages 165, 166.

1. $m=5, n=-1$; $m=-1, n=5$; $m=-1, n=-4$; $m=-4, n=-1$.

2. $x=0, y=0$; $x=5, y=-5$; $x=5, y=4$; $x=1\frac{1}{2}, y=\frac{1}{2}$.

3. $A=3, B=1$; $A=-1, B=-3$; $A=1, B=3$; $A=-3, B=-1$.

4. $p=-\frac{1}{4}, q=796$.

5. $x=3, y=6$; $x=6, y=3$; $x=-\frac{19}{2}+\frac{1}{2}\sqrt{513}, y=-\frac{19}{2}-\frac{1}{2}\sqrt{513}$;
$x=-\frac{19}{2}-\frac{1}{2}\sqrt{513}, y=-\frac{19}{2}+\frac{1}{2}\sqrt{513}$.

6. $r=5, s=-1$; $r=1, s=-5$; $r=1, s=4$; $r=-4, s=-1$.

7. $M=4, N=5$; $M=5, N=4$; $M=-\frac{9}{2}-\frac{1}{2}\sqrt{161}, N=-\frac{9}{2}+\frac{1}{2}\sqrt{161}$;
$M=-\frac{9}{2}+\frac{1}{2}\sqrt{161}, N=-\frac{9}{2}-\frac{1}{2}\sqrt{161}$.

8. $x=0, y=0$; $x=4, y=1$; $x=1, y=-2$; $x=-\frac{1}{2}, y=-\frac{1}{2}$.

9. $w=2, v=1$; $w=1, v=2$; $w=-1, v=-2$; $w=-2, v=-1$;
$w=\frac{1}{2}\sqrt{-1}+\frac{1}{2}\sqrt{11}, v=\frac{1}{2}\sqrt{-1}-\frac{1}{2}\sqrt{11}$; $w=\frac{1}{2}\sqrt{-1}-\frac{1}{2}\sqrt{11}$,
$v=\frac{1}{2}\sqrt{-1}+\frac{1}{2}\sqrt{11}$; $w=-\frac{1}{2}\sqrt{-1}+\frac{1}{2}\sqrt{11}, v=-\frac{1}{2}\sqrt{-1}-\frac{1}{2}\sqrt{11}$;
$w=-\frac{1}{2}\sqrt{-1}-\frac{1}{2}\sqrt{11}, v=-\frac{1}{2}\sqrt{-1}+\frac{1}{2}\sqrt{11}$.

10. $d=13,\ t=\frac{1}{3}\sqrt{33}$; $d=-13,\ t=\frac{1}{3}\sqrt{33}$; $d=13,\ t=-\frac{1}{3}\sqrt{33}$; $d=-13,\ t=-\frac{1}{3}\sqrt{33}$.

11. $P=2,\ Q=3$. The other three solutions may be obtained approximately graphically.

12. $a=0,\ b=0$; $a=-\frac{1}{3},\ b=-\frac{1}{2}$.

13. $h=0,\ k=0$; $h=14,\ k=14$; $h=1-\sqrt{13},\ k=1+\sqrt{13}$; $h=1+\sqrt{13},\ k=1-\sqrt{13}$.

14. $x=25$; $y=9$; $x=9,\ y=25$; $x=-81+8\sqrt{-97},\ y=-81-8\sqrt{-97}$; $x=-81-8\sqrt{-97},\ y=-81+8\sqrt{-97}$.

15. $m=27,\ n=8$; $m=-8,\ n=-27$. **16.** Impossible. **17.** $x=10,\ y=6$.

18. $w=8,\ v=2$; $w=2,\ v=8$. **19.** $x=\frac{1}{3},\ y=\frac{1}{2}$; $x=-\frac{1}{2},\ y=-\frac{1}{3}$.

Pages 167, 168.

1. $\frac{3}{4}$. **2.** $\frac{3}{100}$; 0.03. **3.** $\frac{1}{25000}$; 0.00004.

4. 3.1416, approx.; no; it cannot be expressed by two whole numbers.

5. $\frac{36}{5}$; 7.2. **7.** $\frac{4}{5}$. **9.** Ratio $=\frac{25}{36}$.

6. $\frac{1}{m^2-mn+n^2}$. **8.** $\frac{vv'}{tt'}$. **12.** Yes.

13. Ratio $=\sqrt{2}$. **14.** $\frac{5}{8}$. **15.** \$960; \$1440.

Pages 169, 172.

1. 3.
2. $\frac{2}{3}$.
3. $3\frac{3}{8}$.
4. $13\frac{1}{3}$.
5. 21.
6. $10\frac{1}{2}$.
7. $3\frac{3}{5}$.
8. $110\frac{1}{4}$.
9. $\frac{xz}{y}$.
10. $\frac{v8}{2t}$.
11. k^2.
12. $\frac{a^2}{bc}$.
13. 0.
14. $\frac{r^2}{r'-r}$.
15. 8.
16. $2\frac{2}{3}$.
17. 10.
18. $\frac{1}{16}$.
19. $2x^3$.
20. m^2-n^2.
21. 48.
22. $\frac{1}{8}$.
23. $\frac{1}{27}$.
24. wv^2.
25. $\frac{q^2}{p}$.
26. $(a+b)(a^2-b^2)$.
27. ± 4.
28. ± 15.
29. ± 18.
30. $\pm\sqrt{35}$.

31. $\pm\sqrt{ab}$. **33.** 7. **35.** $4\frac{2}{3}$ inches.

32. $\pm(V-V^2)$. **34.** 15,000. **36.** $26\frac{2}{3}$ in.; $33\frac{1}{3}$ in.

37. $42\frac{6}{17}$ ft.
38. $10\frac{2}{7}$ in.; $7\frac{5}{7}$ in.
39. $8\frac{2}{5}$ in.; $5\frac{3}{5}$ in.
40. 9.9 − in.; 6.1 + in.
41. 11.1 + ft.; 6.9 − ft.
42. $5\frac{1}{3}$ ft.
43. 58 ft.
44. 80 lb.
45. $42\frac{6}{17}$ lb.
46. 240 lb.
47. 54 lb.
48. 90 lb.

Pages 178, 181.

1. $\frac{v}{t}=k$; $\frac{v'}{t'}=\frac{v''}{t''}$.
2. $v \propto \frac{1}{p}$; $vp=k$; $v'p'=v''p''$.
3. 45. **4.** $\frac{1}{15}$. **5.** 900. **6.** 56 lb. **7.** 90. **8.** 72.
9. $157\frac{1}{2}$ lb.; 90 lb.; 45 lb.
10. 96 lb.; $37\frac{1}{2}$ lb.; $12\frac{12}{49}$ lb.
11. 256 ft.; 1600 ft.; 57,600 ft.
12. 2400 lb.
13. 4320 lb.
14. $\frac{1}{2}$ candle-power.
15. 4 candle-power.
16. 75 candle-power.
17. 2 ft.
18. $\frac{9}{10}$ sec.; $\frac{4}{5}$ sec.; $\frac{2}{5}$ sec.
19. 400 cm.; 1600 cm.; 10,000 cm.
20. $355\frac{5}{9}$ gal.
21. 88 days.
22. 4333 days.

Page 183.

1. Circumference. **2.** 1. **3.** $\frac{1}{3}$. **4.** 0; 0; 6. **5.** 0. **6.** 0.

Page 187.

1. $\frac{1}{2}$. **2.** $\frac{1}{12}$. **3.** −4. **4.** $\frac{5}{7}$.
5. Inconsistent.
6. Inconsistent.
7. Inconsistent.
8. Indeterminate.
9. Indeterminate.
10. Indeterminate.

Pages 187, 189.

5. $\pm(m^3-n^3)$.
6. $2\sqrt{3}$.
7. $F=kw$.
8. $F=\frac{kmm'}{d^2}$.
9. $F=\frac{kwv^2}{l}$.
10. $Hd^2=k$.
11. $R=\frac{kl}{A}$.
12. 0.
13. 0.
14. 0.
15. 8.
16. 8.
17. 1.
18. 0.
19. $\frac{1}{4}$.
21. 1.
22. 2.
23. $\frac{1}{2}$.
24. $\frac{1}{3}$.
25. 1.
26. $\frac{1}{2}$.
27. 1.
28. 6.

Page 190.

1. 5, 6, 7, 8.
2. 10, 12, 14, 16.
3. 11, 8, 5, 2.
4. 16, 32, 64, 128.
5. $\frac{1}{81}, \frac{1}{243}, \frac{1}{729}, \frac{1}{2187}$.
6. 25, 36, 49, 64.
7. $-12, -17, -22, -27$.
8. $\frac{5}{6}, \frac{6}{7}, \frac{7}{8}, \frac{8}{9}$.
9. 41, 122, 365, 1094.
10. $16n^4, 32n^5, 64n^6, 128n^7$.
11. $\sqrt{14}, \sqrt{17}, \sqrt{20}, \sqrt{23}$.
12. $-32a^3t^{10}, 16a^4t^8, -8a^5t^6, 4a^6t^4$.

Pages 192, 193.

1. 146.
2. 61.
3. 61.
4. -102.
5. $15\frac{1}{2}$.
6. 257.
7. 16th.
8. 31st.
9. 37th.
10. 3.
11. 5.
12. 8.
13. 57.
14. 2; 3.
15. $-7\frac{1}{3}$; $3\frac{2}{3}$.
16. 49; 81.
17. $14\frac{1}{2}$; $26\frac{1}{2}$.
18. 200.
19. 199.
20. 496.
21. $73n$.
22. $1+45x$.
23. $a+(n-2)d$.
24. $a+(n-4)d$.
25. 624 ft.; 1904 ft.
26. $at-\frac{1}{2}a$.
27. 14 sec.

Pages 194, 195.

1. 5, 9, 13, 17, 21, 25.
2. 12, $8\frac{1}{2}$, 5, $1\frac{1}{2}$, -2.
3. 3, 7, 11, 15, 19, 23, 27.
4. $-6, -1, 4, 9, 14, 19, 24, 29$.
5. $\frac{1}{2}, -1, -2\frac{1}{2}, -4, -5\frac{1}{2}, -7, -8\frac{1}{2}, -10$.
6. $1, 1\frac{4}{13}, 1\frac{8}{13}, 1\frac{12}{13}, 2\frac{3}{13}, 2\frac{7}{13}, 2\frac{11}{13}, 3\frac{2}{13}, 3\frac{6}{13}$, etc.
7. $3, 5\frac{1}{7}, 7\frac{2}{7}, 9\frac{3}{7}, 11\frac{4}{7}, 13\frac{5}{7}, 15\frac{6}{7}, 18, 20\frac{1}{7}$, etc.
8. $\frac{1}{2}, \frac{4}{9}, \frac{7}{18}, \frac{1}{3}$.
9. $n, \frac{31}{36}n, \frac{13}{18}n, \frac{7}{12}n, \frac{4}{9}n, \frac{11}{36}n, \frac{1}{6}n$.
10. 12; 40; 25; 2; -2; $\frac{13}{16}$; $\frac{1}{12}$; $\frac{3}{2}\sqrt{3}$; x.
11. a^2+b^2.
12. 48 ft. per second.
13. 176 ft. per second.
14. 32 mi. per hour; $\frac{5}{8}$ hr.
15. $12\frac{4}{5}$ in.; $23\frac{3}{5}$ in.; $34\frac{2}{5}$ in.; $45\frac{1}{5}$ in.
16. 3 oz.; 5 oz.; 7 oz.; 9 oz.; 11 oz.; 13 oz.
17. 4°.

Pages 197, 199.

1. 1575.
2. 608.
3. 231.
4. -945.
5. $117\frac{1}{2}$.
6. -2385.
7. 5050.
8. 10,100.
9. 10,000.
10. 2475.
11. 7350.
12. 7500.
14. n^2+n.
15. 80.
16. 5 or 6.

17. 24.
18. 20.
19. $n=13$; $S=286$.
20. $n=24$; $l=117$.
21. $a=42$; $l=-26$.
22. −8, −4, 0, 4, 8.
23. 156.
24. 300.
25. 2550 yd.
26. 150 in.
27. 590 ft.; 30 sec.
28. 576 ft.; 1600 ft.; 57,600 ft.
29. 16,896 ft.
30. 10 sec.
31. $10,600; $7300.
32. $12,250.
33. 144; 121; 100; 650.
34. 4994 − in.
35. 2, 6, 10.
36. 1, 5, 9.

Pages 201, 202.

1. 128.
2. 100,000,000.
3. $\frac{16}{81}$.
4. 1458.
5. 1024.
6. $-\frac{3}{8}$.
7. 16,384.
8. 2.
9. 6.
10. $\frac{1}{1024}$.
11. $\pm 12\sqrt{2}$.
12. $\frac{27}{4}$.
13. 9, 18, 36, 72, 144; or −9, 18, −36, 72, −144.
14. $\frac{1}{128}$, $\frac{1}{32}$, $\frac{1}{8}$, $\frac{1}{2}$, 2, 8.
15. 16.
16. $\frac{27\sqrt{2}}{4}$.
17. $\frac{n^2}{m}$, $\frac{n^3}{m^2}$.
18. $11,809.80.
19. $119.41.
20. $579.64.
21. 259,374.
22. $\frac{15}{16384}$ ft.

Pages 203, 204.

1. 7, 14, 28, 56.
2. 2, 6, 18, 54.
3. 16, $8\sqrt[4]{54}$, $4\sqrt{54}$, $2\sqrt[4]{2916}$, 54.
4. 1, 4, 16, 64, 256, 1024.
5. 80, 40, 20, 10, 5.
6. 24, 16, $\frac{32}{3}$, $\frac{64}{9}$, $\frac{128}{27}$, $\frac{256}{81}$.
7. −9, −18, −36, −72, −144.
8. 1, $\sqrt[3]{2}$, $\sqrt[3]{4}$, 2.
9. m, $\sqrt[3]{m^2n}$, $\sqrt[3]{mn^2}$, n.
10. ±2; ±12; ±10; $\pm 2\sqrt{3}$; ±4.
11. $\pm(a^2-ab)$. 12. 15; $4\sqrt{3}$. 13. 4; 16. 14. 10; 18. 15. 11.088; 23.088.

Pages 205, 206.

1. 6138.
2. 364.
3. 86.
4. $15\frac{15}{16}$.
5. $63\frac{63}{64}$.
6. 5555.
7. $\frac{1-x^{10}}{1-x}$.
8. $49\frac{255}{256}$.
9. $\frac{6n-384n^7}{1+2n}$.
10. $24+26\sqrt{3}$.
11. $2\frac{85}{128}$.
12. 3.
13. $\frac{2}{3}$; $\frac{4}{81}$.
14. 8.
15. 6.
16. $149\frac{23}{25}$ ft.
17. $152\frac{229}{576}$ cu. in.
18. 4 yr.
19. $\frac{31}{32}$.
20. $1.125509 −.
21. $1159.27 +.
22. $119.41 −.
23. $3736.29.
24. $18,681.45.

Pages 208, 209.

1. 16. **2.** 18. **3.** $22\frac{1}{2}$. **4.** $3\frac{3}{11}$. **5.** $4\frac{1}{2}$. **6.** $\frac{2}{3}$. **7.** $1\frac{1}{4}$. **8.** $4+2\sqrt{2}$.

9. $\frac{1}{1-n}$. **10.** $\frac{n^2}{n^2-1}$. **11.** 60. **12.** $-\frac{1}{16}$.

13. $\frac{3}{2}, \frac{3}{8}, \frac{3}{32}, \frac{3}{128}, \frac{3}{512}$; or $\frac{1}{2}, \frac{3}{8}, \frac{9}{32}, \frac{27}{128}, \frac{81}{512}$. **14.** $\frac{2}{27}$ or $4\frac{20}{27}$.

15. $\frac{2}{3}$. **17.** $\frac{13}{33}$. **19.** $\frac{50}{333}$. **21.** $4\frac{4}{11}$.

16. $\frac{8}{11}$. **18.** $\frac{5}{11}$. **20.** $1\frac{1}{3}$. **22.** $3\frac{238}{330}$.

23. $\frac{51}{1100}$. **24.** 12π in. **25.** $\frac{16}{3}\pi$ sq. in.

Pages 209, 210.

2. $\frac{1}{31}$. **3.** $\frac{1}{49}$. **4.** $\frac{2}{25}$. **5.** $\frac{1}{89}$. **6.** $\frac{2}{111}$.

7. $-\frac{2}{3}, -2, 2, \frac{2}{3}, \frac{2}{5}$. **10.** $\frac{1}{4}, \frac{18}{61}, \frac{9}{25}, \frac{6}{13}, \frac{9}{14}, \frac{18}{17}, 3$.

8. $\frac{1}{9}$; 3. **13.** ± 12.

9. $-\frac{6}{5}, -2, -6, 6, 2, \frac{6}{5}$. **14.** 0.1 and 159.9, approx.

Page 213.

1. n^{10}.
2. V^{12}.
3. k^9.
4. R^{15}.
5. 2^8.
6. 5^7.
7. 3^9.
8. $(-1)^5$.
9. $(-2)^6$.
10. $(-a)^8$.
11. x^{2n}.
12. t^{n+2}.
13. P^{2m}.
14. a^7.
15. y^4.
16. D^8.
17. r.
18. mv^2.
19. $2t$.
20. 2^3.
21. 3^2.
22. $(-1)^2$.
23. $(-n)^4$.
24. x^{20}.
25. N^{18}.
26. s^{48}.
27. k^{100}.
28. 2^6.
29. 3^8.
30. $(-1)^{18}$.
31. $(-b)^{20}$.
32. x^{n^2-n}.
33. m^4n^4.
34. $a^{12}b^{18}$.
35. $x^{15}y^3$.
36. $v^{30}t^{18}$.
37. $2^3\times 5^3$.
38. $7^2\times 3^2$.
39. $a^{5n}b^{10n}$.
40. $\frac{3^3}{4^3}$.
41. $\frac{2^4}{3^4}$.
42. $\frac{4^4}{5^4}$.
43. $+\frac{1^6}{2^6}$.
44. $-\frac{2^3}{3^3}$.
45. $\frac{a^4}{b^4}$.
46. $\frac{8m^6}{27n^{15}}$.
47. $\frac{v^{15}t^{25}}{32a^{20}s^{10}}$.
48. a^3.
49. n^3.
50. x^2y^4.
51. $2r^4$.
52. a^2b^3.
53. a^2.
54. t^2.
55. m^6.
56. $1024a^{20}b^4$.

Page 214.

1. 1.
2. 1.
3. 1.
4. 1.
5. 1.
6. 81.
7. n^5.
8. 1.

Pages 215, 216.

1. $\frac{1}{4}$.
2. $\frac{1}{32}$.
3. $\frac{1}{81}$.
4. $\frac{1}{7}$.
5. $\frac{1}{1000000}$.
6. 1.
7. 4.
8. 1,000,000.
9. 4.
10. $\frac{9}{4}$.
11. 125.
12. $\frac{64}{9}$.
13. $\frac{625}{16}$.
14. $\frac{1}{16}$.
15. $\frac{9}{16000}$.
16. 324.
17. $\frac{1}{n^4}$.
18. $\frac{a^2}{b^3}$.
19. $\frac{\pi R^2}{r^2}$.
20. $\frac{n^4}{m^2}$.
21. a^5b^5.
22. $\frac{1}{ab^2c^3}$.
23. $\frac{2}{-x^2y^2}$.
24. $\frac{-5N^3}{-3P^4}$.
25. $\frac{1}{x^2y^4z^6}$.
26. $\frac{x^3}{y^3}$.
27. $\frac{b}{-6a^6}$.
28. $\frac{1}{5V^2}$.
29. $\frac{m}{n}+\frac{n}{m}$.
30. $\frac{b+a}{b-a}$.
31. $\frac{x^2y^2-1}{x^2y^2+1}$.
32. $\frac{a-b}{a+b}$.
33. N^6.
34. a^2b^{-2}.
35. $\frac{2}{3}xy^{-1}$.
36. $-\frac{1}{5}n^{-1}$.
37. $v^{-2}t^2$.
38. $x^{-1}+y^{-1}$.
39. $ab^{-1}+a^{-1}b$.
40. $2mn$.
41. $(a^2+ab+b^2)^{-1}$.

Page 217.

1. 2.
2. 3.
3. 9.
4. 8.
5. 125.
6. 8.
7. 27.
8. 25.
9. 8.
10. 32.
11. 243.
12. 256.
13. $\frac{1}{2}$.
14. $\frac{1}{8}$.
15. $\frac{1}{9}$.
16. $\frac{1}{64}$.
17. $\frac{9}{4}$.
18. $\frac{256}{625}$.
19. $\frac{16}{9}$.
20. $\frac{243}{32}$.
21. $\frac{4}{27}$.
22. $\frac{8}{3125}$.
23. $\frac{9}{5}$.
24. 24.
25. $\frac{1}{2}$.
26. $\sqrt[3]{a^2b}$.
27. $\frac{\sqrt{q}}{\sqrt{p}}$.
28. $\frac{1}{\sqrt[12]{M^8V^9}}$.
29. $\frac{4}{\sqrt{x^5}}$.
30. $-9\sqrt[4]{t^3}$.
31. $\frac{2\sqrt{m^3}}{\sqrt[3]{n^2}}$.
32. $\frac{9\sqrt[5]{t^2}}{\sqrt[3]{V^4}}$.
33. $\frac{z}{\sqrt{xy}}$.
34. $\frac{\sqrt[3]{p^4q^9}}{m^2}$.
35. $\frac{\sqrt[4]{(a+b)^3}}{\sqrt[4]{(a-b)^3}}$.
36. $h-k$.
37. $\sqrt{mn}$.
38. $v^{\frac{2}{5}}$.
39. $n^{\frac{7}{3}}$.
40. $a^{\frac{3}{2}}$.
41. $w^{\frac{1}{2}}t^{\frac{1}{3}}$.
42. $a^{\frac{1}{2}}b^{\frac{3}{4}}m^{\frac{1}{4}}n^{\frac{1}{4}}$.
43. $3^{\frac{1}{2}}x^{\frac{1}{2}}y$.
44. $b^{\frac{3}{4}}c^{\frac{1}{2}}$.
45. $a^{\frac{1}{m}}$.
46. $x^{\frac{m}{2}}$.
47. $(a-b)^{\frac{1}{2}}$.
48. $(1+x^2)^{\frac{1}{3}}$.
49. $x^{\frac{n}{m}}x^{\frac{m}{n}}$.
50. $(1-x)^{-\frac{1}{2}}$.
51. $(1+x^2)^{-\frac{2}{3}}$.
52. $(a+b)^{\frac{3}{2}}(a-b)^{-\frac{3}{2}}$.

Page 219.

1. A^2.
2. $4n$.
3. 1.
4. $12t^4$.
5. a.
6. $15p^{-\frac{1}{6}}$.
7. $h^{\frac{1}{5}}k^{\frac{1}{6}}$.
8. $V^{-\frac{1}{12}}t^{-\frac{1}{16}}$
9. $y^{\frac{5}{4}}z^{\frac{7}{12}}$.
10. x^2.
11. $a^{-2}n^{-6}$.
12. w^3v^{-2}.
13. $p^{-4}q^{-4}r^{-4}$.
14. $N^{\frac{3}{5}}$.
15. $K^{\frac{1}{3}}$.
16. ab.
17. $x^{-\frac{1}{3}}yz^{-\frac{5}{4}}$.
18. $2P^{-\frac{5}{4}}V^{-\frac{1}{7}}$.
19. a^4b^6.
20. $x^{-20}y^5$.
21. $\pi^{-4}R^{-8}$.
22. $m^{-1}n^{-3}$.
23. $2^{\frac{2}{3}}H^{\frac{1}{2}}n^{-\frac{2}{3}}$.
24. $\frac{1}{125}M^{\frac{1}{2}}A^{-1}$.
25. $\frac{1}{8}xy^{\frac{3}{8}}z^{\frac{3}{2}}$.
26. $3^{-\frac{1}{3}}a^{-1}b^{\frac{1}{2}}c^{\frac{3}{4}}$.
27. $\frac{1}{9}p^6d^{\frac{1}{3}}$.
28. $\frac{1}{8}rs^{-\frac{6}{11}}$.
29. $a^{\frac{5}{6}}$.
30. $t^{\frac{25}{12}}$.
31. $n^{\frac{5}{4}}$.
32. k.
33. a^2x.
34. $a^{\frac{5}{6}}$.
35. $w^{-\frac{5}{12}}v^{-\frac{1}{6}}$.
36. $n^{\frac{1}{3}}m^{-\frac{1}{6}}$.
37. $k^{-\frac{7}{6}}n^{-\frac{4}{3}}$.
38. $a^{-\frac{11}{3}}b^4$.
39. $t^{-\frac{112}{39}}$.
40. $n^{\frac{2}{3}}$.
41. $x^{-\frac{1}{4}}$.
42. $x-y$.
43. W^3-V^{-3}.
44. $m^{\frac{1}{2}}-n^{\frac{1}{2}}$.
45. $R^{\frac{2}{3}}-R^{\frac{1}{3}}r^{\frac{1}{3}}+r^{\frac{2}{3}}$.
46. $x^{\frac{4}{3}}+x^{\frac{2}{3}}y^{\frac{2}{3}}+y^{\frac{4}{3}}$.
47. $x+x^{\frac{1}{2}}-2$.
48. $9m^{-\frac{4}{5}}-16n^6$.
49. $a^{\frac{1}{2}}+2b^{\frac{1}{2}}$.
50. $t^{-2}+t^{-1}+1$.
51. $1-y^{-\frac{1}{3}}+y^{-\frac{2}{3}}$.
52. $n^{-2}-3n^{-1}+9$.
53. $k^{-3}+k^{-2}+k^{-1}+1$.
54. $v^{\frac{8}{5}}-v^{\frac{6}{5}}t^{\frac{2}{5}}+v^{\frac{4}{5}}t^{\frac{4}{5}}-v^{\frac{2}{5}}t^{\frac{6}{5}}+t^{\frac{8}{5}}$.
55. $M^{\frac{4}{3}}+2M^{\frac{2}{3}}N^{-\frac{3}{4}}+N^{-\frac{3}{2}}$.
56. $a-2a^{\frac{1}{2}}b^{-\frac{1}{2}}+b^{-1}$.
57. $1-2x^{-\frac{1}{4}}+x^{-\frac{1}{2}}$.
58. $E^{-\frac{1}{2}}+2E^{-\frac{1}{4}}F^{-\frac{1}{4}}+F^{-\frac{1}{2}}$.
59. $e^{2x}+E^{-2x}+6-4e^x-4e^{-x}$.
60. $1-2x^{-1}+x^{-2}+2x^{-3}-2x^{-4}+x^{-6}$.
61. $n^{\frac{1}{2}}+2n^{\frac{1}{4}}+3+2n^{-\frac{1}{4}}+n^{-\frac{1}{2}}$.
62. $h^{\frac{3}{4}}+h^{\frac{1}{2}}k^{\frac{1}{4}}+h^{\frac{1}{4}}k^{\frac{1}{2}}+k^{\frac{3}{4}}$.
63. $A^{-n}-2A^{-\frac{1}{2}n}+1$.
64. $a-b$.
65. $x^{\frac{8}{3}}+x^{\frac{4}{3}}y^{\frac{4}{3}}+y^{\frac{8}{3}}$.
66. t^2-1.
67. n^4-1+n^{-4}.
68. $p^{\frac{4}{5}}+4p^{\frac{2}{5}}q^{\frac{2}{5}}+16q^{\frac{4}{5}}$.
69. m^2-n^2.
70. $E+F$.
71. $x^{\frac{1}{2}}+1+x^{-\frac{1}{2}}$.
72. $\frac{2}{\sqrt[3]{n^2}}-3$.

Pages 220, 221.

1. $a^{-2n}+a^{-n}+1.$
2. $x^n+y^n.$
3. $t^{-4n}-2t^{-3n}+3t^{-2n}-4t^{-n}+5-4t^n+3t^{2n}-2t^{3n}+t^{4n}.$
4. $m^{\frac{1}{3}}+n^{\frac{1}{3}}.$
5. $(4+n^{-1})(4-n^{-1}).$
6. $(R+R^{-1})^2.$
7. $(3-N^{-1})(9+3N^{-1}+N^{-2}).$
8. $(k^{-1}-3)(k^{-1}+2).$
9. $3x^{\frac{1}{2}}-2+x^{-\frac{1}{2}}.$
10. $m-3m^{\frac{1}{2}}n^{\frac{1}{3}}+3n^{\frac{2}{3}}.$
11. $A+4-2A^{-1}.$
12. $h^2+h^{-1}k+hk^{-1}.$
13. $r^{\frac{2}{5}}+2r^{\frac{1}{5}}+r^{-\frac{1}{5}}.$
14. $a^{-\frac{2}{3}}-a^{-\frac{1}{3}}b^{\frac{1}{3}}+b^{\frac{2}{3}}.$

Page 224.

1. $a^7+7a^6b+21a^5b^2+35a^4b^3+35a^3b^4+21a^2b^5+7ab^6+b^7.$
2. $a^7-7a^6b+21a^5b^2-35a^4b^3+35a^3b^4-21a^2b^5+7ab^6-b^7.$
3. $m^8+8m^7n+28m^6n^2+56m^5n^3+70m^4n^4+56m^3n^5+28m^2n^6+8mn^7+n^8.$
4. $m^8-8m^7n+28m^6n^2-56m^5n^3+70m^4n^4-56m^3n^5+28m^2n^6-8mn^7+n^8.$
5. $x^9+9x^8y+36x^7y^2+84x^6y^3+126x^5y^4+126x^4y^5+84x^3y^6+36x^2y^7+9xy^8 +y^9.$
6. $v^{10}-10v^9t+45v^8t^2-120v^7t^3+210v^6t^4-252v^5t^5+210v^4t^6-120v^3t^7+45v^2t^8 -10vt^9+t^{10}.$
7. $h^{12}+12h^{11}k+66h^{10}k^2+220h^9k^3+495h^8k^4+792h^7k^5+924h^6k^6+792h^5k^7 +495h^4k^8+220h^3k^9+66h^2k^{10}+12hk^{11}+k^{12}.$
8. $P^{13}-13P^{12}Q+78P^{11}Q^2-286P^{10}Q^3+715P^9Q^4-1287P^8Q^5+1716P^7Q^6 -1716P^6Q^7+1287P^5Q^8-715P^4Q^9+286P^3Q^{10}-78P^2Q^{11}+13PQ^{12}-Q^{13}.$
9. $1+4x+6x^2+4x^3+x^4.$
10. $1-5n+10n^2-10n^3+5n^4-n^5.$
11. $a^6+6a^5+15a^4+20a^3+15a^2+6a+1.$
12. $32x^5+80x^4y+80x^3y^2+40x^2y^3+10xy^4+y^5.$
13. $a^{16}-8a^{14}b^2+28a^{12}b^4-56a^{10}b^6+70a^8b^8-56a^6b^{10}+28a^4b^{12}-8a^2b^{14}+b^{16}.$
14. $M^6-18M^5N^2+135M^4N^4-540M^3N^6+1215M^2N^8-1458MN^{10}+729N^{12}.$
15. $64V^6+576V^5t+2160V^4t^2+4320V^3t^3+4860V^2t^4+2916Vt^5+729t^6.$
16. $1-8x^3+24x^6-32x^9+16x^{12}.$

17. $128p^{14}+448p^{12}q+672p^{10}q^2+560p^8q^3+280p^6q^4+84p^4q^5+14p^2q^6+q^7$.

18. $1-15V^3+90V^6-270V^9+405V^{12}-243V^{15}$.

19. $y^{30}+10y^{27}z^3+45y^{24}z^6+120y^{21}z^9+210y^{18}z^{12}+252y^{15}z^{15}+210y^{12}z^{18}+120y^9z^{21}$ $+45y^6z^{24}+10y^3z^{27}+z^{30}$.

20. $1+16b^4+112b^8+448b^{12}+1120b^{16}+1792b^{20}+1792b^{24}+1024b^{28}+256b^{32}$.

21. $1-3n+\frac{15}{4}n^2-\frac{5}{2}n^3+\frac{15}{16}n^4-\frac{3}{16}n^5+\frac{1}{64}n^6$.

22. $\frac{1}{64}E^{12}+\frac{1}{16}E^{10}F^2+\frac{5}{48}E^8F^4+\frac{5}{54}E^6F^6+\frac{5}{108}E^4F^8+\frac{1}{81}E^2F^{10}+\frac{1}{729}F^{12}$.

23. $\frac{32}{243}a^{20}-\frac{80}{81}a^{16}+\frac{80}{27}a^{12}-\frac{40}{9}a^8+\frac{10}{3}a^4-1$.

24. $1296R^8+432R^6+54R^4+3R^2+\frac{1}{16}$.

25. $a^{18}+18a^{17}b+153a^{16}b^2+816a^{15}b^3+\ldots\ldots.$

26. $a^{20}-20a^{19}b+190a^{18}b^2-1140a^{17}b^3+\ldots\ldots.$

27. $1+40N+780N^2+9880N^3+\ldots\ldots.$

28. $t^{30}-30t^{29}+435t^{28}-4060t^{27}+\ldots\ldots.$

29. $x^{20}+40x^{19}+760x^{18}+9120x^{17}+\ldots\ldots.$

30. $M^{48}-48M^{46}N+1104M^{44}N^2-16192M^{42}N^3+\ldots\ldots.$

31. $w^{44}-66w^{42}v^2+2079w^{40}v^4-41580w^{38}v^6+\ldots\ldots.$

32. $(\frac{1}{2}x)^{30}+30(\frac{1}{2}x)^{29}(2y)+435(\frac{1}{2}x)^{28}(2y)^2+4060(\frac{1}{2}x)^{27}(2y)^3+\ldots\ldots.$

33. $n^{40}-80n^{38}+3120n^{36}-79040n^{34}+\ldots\ldots.$

34. $\frac{65536a^{16}}{b^{16}}+\frac{131072a^{13}}{b^{13}}+\frac{122880a^{10}}{b^{10}}+\frac{71680a^7}{b^7}+\ldots\ldots.$

35. $a^n-na^{n-1}b+\frac{n(n-1)}{2}a^{n-2}b^2-\frac{n(n-1)(n-2)}{6}a^{n-3}b^3$

$+\frac{n(n-1)(n-2)(n-3)}{24}a^{n-4}b^4-\ldots\ldots.$

36. 3.14−. **37.** 6.19+. **38.** 1.37−. **39.** 0.00+. **40.** 59604.64+.

41.

1
1 2 1
1 3 3 1
1 4 6 4 1
1 5 10 10 5 1
1 6 15 20 15 6 1
1 7 21 35 35 21 7 1
1 8 28 56 70 56 28 8 1
1 9 36 84 126 126 84 36 9 1
1 10 45 120 210 252 210 120 45 10 1

Page 226.

1. $1-n+n^2-n^3+n^4-\ldots\ldots.$
2. $1+3t+6t^2+10t^3+15t^4+\ldots\ldots.$
3. $1+\frac{1}{2}x-\frac{1}{8}x^2+\frac{1}{16}x^3-\frac{5}{128}x^4+\ldots\ldots.$
4. $1-\frac{1}{3}R-\frac{1}{9}R^2-\frac{5}{81}R^3-\frac{10}{243}R^4-\ldots\ldots.$
5. $1+\frac{1}{4}a-\frac{3}{32}a^2+\frac{7}{128}a^3-\frac{77}{2048}a^4+\ldots\ldots.$
6. $1+x^3+x^6+x^9+x^{12}+\ldots\ldots.$
7. $1-2k^2+3k^4-4k^6+5k^8-\ldots\ldots.$
8. $1-\frac{1}{2}P^2+\frac{3}{8}P^4-\frac{5}{16}P^6+\frac{35}{128}P^8-\ldots\ldots.$
9. $1-2a^2+6a^4-20a^6+70a^8-\ldots\ldots.$
10. $1+3y^3+18y^6+126y^9+945y^{12}+\ldots\ldots.$
11. $a^{-\frac{1}{2}}+\frac{1}{4}a^{-\frac{5}{2}}b^2+\frac{5}{32}a^{-\frac{9}{2}}b^4+\frac{15}{128}a^{-\frac{13}{2}}b^6+\frac{195}{2048}a^{-\frac{17}{2}}b^8+\ldots\ldots.$
12. $x^{\frac{1}{4}}+\frac{1}{2}x^{-\frac{1}{4}}y^{\frac{1}{2}}-\frac{1}{8}x^{-\frac{3}{4}}y+\frac{1}{16}x^{-\frac{5}{4}}y^{\frac{3}{2}}-\frac{5}{128}x^{-\frac{7}{4}}y^2+\ldots\ldots.$
13. $m^{\frac{1}{4}}+\frac{1}{2}m^{\frac{3}{4}}n^{-\frac{1}{2}}+\frac{3}{8}m^{\frac{5}{4}}n^{-1}+\frac{5}{16}m^{\frac{7}{4}}n^{-\frac{3}{2}}+\frac{35}{128}m^{\frac{9}{4}}n^{-2}+\ldots\ldots.$
14. $2^{\frac{2}{3}}v^{\frac{4}{9}}+\frac{2}{3}\times 2^{-\frac{1}{3}}v^{-\frac{2}{9}}t^{\frac{3}{2}}-\frac{1}{9}\times 2^{-\frac{4}{3}}v^{-\frac{8}{9}}t^3+\frac{4}{81}\times 2^{-\frac{7}{3}}v^{-\frac{14}{9}}t^{\frac{9}{2}}-\frac{7}{243}\times 2^{-\frac{10}{3}}v^{-\frac{20}{9}}t^6$
$+\ldots\ldots.$
15. $3^{-\frac{1}{3}}p^{\frac{1}{4}}+\frac{2}{3}\times 3^{-\frac{4}{3}}pq^{-1}+\frac{5}{9}\times 3^{-\frac{7}{3}}p^{\frac{7}{4}}q^{-2}+\frac{112}{81}\times 3^{-\frac{10}{3}}p^{\frac{5}{2}}q^{-3}$
$+\frac{560}{243}\times 3^{-\frac{13}{3}}p^{\frac{13}{4}}q^{-4}+\ldots\ldots.$

Page 227.

1. 1.968−.
2. 3.037−.
3. 4.041+.
4. 2.924+.
5. 5.066−.
6. 6.028−.
7. 2.080+.
8. 4.019+.
9. 2.993−.
10. 2.015+.
11. 4.998+.
12. 3.001−.
13. 3.036+.
14. 1.961−.
15. 2.030+.

Page 228.

1. $20a^6b^6.$
2. $35x^4.$
3. $-77520t^{14}.$
4. $-658008m^{35}n^5.$
5. $220x^{27}y^9.$
6. $A^{-40}B^{18}.$
7. $-\frac{21}{16}x^{18}.$
8. $\frac{88179}{256}V^{-11}.$
9. $\frac{4807}{262144}x^{-\frac{27}{8}}y^{\frac{7}{2}}.$
10. $\frac{54834}{59049}R^{\frac{25}{8}}r^{16}.$
11. $\frac{\sqrt{2}a^{-\frac{5}{2}}b^6}{128}.$
12. $\frac{17\times 35\times 29\times 37\times 41\times 53\times 57}{2\times 4^{20}}a^{-\frac{15}{2}}.$
13. $\frac{21}{256}k^5.$
14. $\frac{33\sqrt{2}n^{14}}{33554432}.$
15. $\frac{27170}{4782969}x^{-\frac{116}{9}}.$
16. $5e^{-5}.$

Page 229.

1. $x^4-8x^{\frac{7}{2}}y^{\frac{1}{2}}+28x^3y-56x^{\frac{5}{2}}y^{\frac{3}{2}}+70x^2y^2-56x^{\frac{3}{2}}y^{\frac{5}{2}}+28xy^3-8x^{\frac{1}{2}}y^{\frac{7}{2}}+y^4$.

2. $a^4+6a^{\frac{10}{3}}b^{\frac{2}{3}}+15a^{\frac{8}{3}}b^{\frac{4}{3}}+20a^2b^2+15a^{\frac{4}{3}}b^{\frac{8}{3}}+6a^{\frac{2}{3}}b^{\frac{10}{3}}+b^4$.

3. $m^{15}+10m^{\frac{27}{2}}n^{\frac{3}{2}}+45m^{12}n^3+120m^{\frac{21}{2}}n^{\frac{9}{2}}+210m^9n^6+252m^{\frac{15}{2}}n^{\frac{15}{2}}+210m^6n^9$
$+120m^{\frac{9}{2}}n^{\frac{21}{2}}+45m^3n^{12}+10m^{\frac{3}{2}}n^{\frac{27}{2}}+n^{15}$.

4. $e^{27}-12e^{\frac{99}{4}}+66e^{\frac{45}{2}}-220e^{\frac{81}{4}}+495e^{18}-792e^{\frac{63}{4}}+924e^{\frac{27}{2}}-792e^{\frac{45}{4}}+495e^9$
$-220e^{\frac{27}{4}}+66e^{\frac{9}{2}}-12e^{\frac{9}{4}}+1$.

5. $a^5b^{-10}+10a^{\frac{7}{2}}b^{-\frac{17}{2}}+45a^2b^{-7}+120a^{\frac{1}{2}}b^{-\frac{11}{2}}+210a^{-1}b^{-4}+252a^{-\frac{5}{2}}b^{-\frac{5}{2}}$
$+210a^{-4}b^{-1}+120a^{-\frac{11}{2}}b^{\frac{1}{2}}+45a^{-7}b^2+10a^{-\frac{17}{2}}b^{\frac{7}{2}}+a^{-10}b^5$.

6. $81m^4n^{-8}+108m^{\frac{7}{2}}n^{-5}+54m^3n^{-2}+12m^{\frac{5}{2}}n+m^2n^4$.

7. $x^3y^{-4}-6x^2y^{-\frac{17}{6}}+15xy^{-\frac{5}{3}}-20y^{-\frac{1}{2}}+15x^{-1}y^{\frac{2}{3}}-6x^{-2}y^{\frac{11}{6}}+x^{-3}y^3$.

8. $a^{\frac{16}{3}}b^{-1}+2a^3b^{-\frac{11}{20}}+\frac{3}{2}a^{\frac{2}{3}}b^{-\frac{1}{10}}+\frac{1}{2}a^{-\frac{5}{3}}b^{\frac{7}{20}}+\frac{1}{16}a^{-4}b^{\frac{4}{5}}$.

9. $1+4x+2x^2-8x^3-5x^4+8x^5+2x^6-4x^7+x^8$.

10. $x^6+3x^4y^2+3x^2y^4+y^6+3x^4z^2+6x^2y^2z^2+3y^4z^2+3x^2z^4+3y^2z^4+z^6$.

11. $16-32m+24m^2-8m^3+m^4+32n-48mn+24m^2n+4m^3n+24n^2-24mn^2$
$+6m^2n^2+8n^3-4mn^3+n^4$.

12. $a^4+b^4+c^4+d^4+6a^2b^2+6a^2c^2+6a^2d^2+6b^2c^2+6b^2d^2+6c^2d^2-4a^3b+4a^3c$
$-4a^3d-12a^2bc+12a^2bd-12a^2cd-4ab^3+12ab^2c-12ab^2d-4b^3c$
$+4b^3d-12b^2cd-12abc^2+4ac^3-12ac^2d-4bc^3+12bc^2d-4c^3d-12abd^2$
$+12acd^2-4ad^3-12bcd^2+4bd^3-4cd^3+24abcd$.

13. $1-3t+6t^2-10t^3+12t^4-12t^5+10t^6-6t^7+3t^8-t^9$.

14. $16a^4+b^4+81c^8+24a^2b^2+216a^2c^4+54b^2c^4-32a^3b+96a^3c^2-144a^2bc^2$
$-8ab^3+72ab^2c^2-12b^3c^2-216abc^4+216ac^6-108bc^6$.

15. $n^{-8}+4n^{-6}+10n^{-4}+16n^{-2}+19+16n^2+10n^4+4n^6+n^8$.

16. $\dfrac{n(n-1)(n-2)\ldots\ldots\text{to } r \text{ factors}}{1\,.\,2\,.\,3\ldots\ldots\text{to } r \text{ factors}}a^{n-r}b^r$.

17. $\dfrac{n(n-1)(n-2)\ldots\ldots\text{to } r-2 \text{ factors}}{1\,.\,2\,.\,3\ldots\ldots\text{to } r-2 \text{ factors}}a^{n-(r-2)}b^{r-2}$.

18. $-56w^{\frac{13}{2}}v^{\frac{11}{2}}$. 19. $210a^{-\frac{8}{5}}b^{-\frac{22}{5}}$. 20. 6.0368+.

Page 230.

2. $\log_2 64 = 6$.
3. $\log_5 25 = 2$.
4. $\log_3 81 = 4$.
5. $\log_4 64 = 3$.
6. $\log_5 125 = 3$.
7. $\log_7 343 = 3$.
8. $\log_{10} 10000 = 4$.
9. $\log_5 15625 = 6$.
10. $\log_6 \frac{1}{36} = -2$.
11. $\log_4 \frac{1}{64} = -3$.
12. $\log_{16} 2 = \frac{1}{4}$.
13. $\log_8 \frac{1}{4} = -\frac{2}{3}$.
15. $2^4 = 16$.
16. $4^2 = 16$.
17. $3^3 = 27$.
18. $10^3 = 1000$.
19. $6^3 = 216$.
20. $4^4 = 256$.
21. $8^{\frac{2}{3}} = 4$.
22. $27^{\frac{4}{3}} = 81$.
23. $7^{-2} = \frac{1}{49}$.
24. $3^{-5} = \frac{1}{243}$.
25. $10^{-3} = 0.001$.
26. $100^{-\frac{3}{2}} = 0.001$.
27. 4.
28. 3.
29. 2.
30. −4.
31. 3.
32. −1.
33. −2.
34. −2.
35. 3.
36. −3.
37. 0.
38. 0.
39. 4.
40. 64.
41. 1024.
42. $\frac{1}{16}$.
43. $\frac{1}{256}$.
44. 2.
45. $\frac{1}{2}$.
46. $\frac{1}{8}$.

Page 233.

1. 1.
2. 2.
3. −2.
4. −4.
5. 0.
6. 4.
7. 0.
8. 0.
9. −1.
10. 3.
11. 1.
12. −3.

Page 236.

1. 2.3324.
2. 2.8280.
3. 2.9731.
4. 2.8555.
5. 0.6646.
6. 1.2989.
7. 2.9191.
8. 1.2041.
9. 1.4624.
10. 0.9294.
11. 2.
12. 2.9542.
13. 0.
14. 0.3010.
15. 3.2263.
16. 1.5349.
17. 2.0022.
18. 2.9670.
19. $\bar{1}.9361$.
20. $\bar{3}.4983$.
21. $\bar{2}.8353$.
22. 3.6302.
23. 0.0398.
24. $\bar{4}.0792$.
25. 1.9999.
26. 3.3079.
27. $\bar{3}.9208$.
28. $\bar{1}.7025$.
29. 4.8363.
30. $\bar{2}.8739$.

Page 237.

1. 472.
2. 54.8.
3. 1210.
4. 6.18.
5. 0.0789.
6. 0.0095.
7. 0.415.
8. 1.69.
9. 57.3.
10. 7270.
11. 1.53.
12. 0.289.
13. 0.0427.
14. 488333.
15. 300.8.
16. 4007.
17. 0.005315.
18. 790.67.
19. 10223.8.
20. 29.53.
21. 0.09474.
22. 7.015.
23. 0.25541.
24. 0.00001052.
25. 1094.5.
26. 0.00029893.
27. 7.4733.
28. 37.445.

Page 239.

1. $\log_a x+\log_a y+\log_a z+\log_a w$.
2. $2\log_a x+2\log_a y$.
3. $\log_a 2+\log_a x+2\log_a y+3\log_a z$.
4. $2\log_a x+\log_a y-3\log_a z$.
5. $\log_a 5+\log_a x-\log_a y-\log_a z-\log_a w$.
6. $\log_a 8+2\log_a x+2\log_a y-3\log_a z-\log_a w$.
7. $2(\log_a x+\log_a y-\log_a z-\log_a w)$.
8. $\frac{2}{3}\log_a x-\frac{3}{4}\log_a y$.
9. $\frac{1}{3}\log_a x+\frac{1}{3}\log_a y-\frac{1}{4}\log_a z$.
10. $\frac{1}{2}\log_a x-\frac{1}{2}\log_a y$.
11. $\frac{1}{2}\log_a x+\frac{1}{2}\log_a y-\log_a z-\frac{1}{2}\log_a w$.
12. $\frac{1}{3}\log_a x+\frac{2}{3}\log_a y-\frac{1}{2}\log_a z-\frac{1}{3}\log_a w$.
13. $\frac{1}{4}\log_a z+\frac{3}{4}\log_a y-\frac{1}{3}\log_a x$.
14. $\frac{1}{2}\log_a x-\frac{1}{2}\log_a y+\frac{1}{2}\log_a z-\frac{1}{2}\log_a w$.
15. $\frac{1}{12}\log_a x-\frac{1}{12}\log_a y$.
16. $\frac{1}{3}\log 1728$.
17. $\frac{1}{2}\log 2+\frac{1}{3}\log 9$.
18. $\frac{1}{2}\log 98-\frac{1}{3}\log 56$.

Pages 241, 243.

1. 23.41.
2. 93.8.
3. 0.00417.
4. 248.43.
5. 25.635.
6. 0.1469.
7. 160.9.
8. 5428.7.
9. 0.08935.
10. 18.279.
11. 0.005164.
12. 9.8825.
13. 7.98.
14. 24.383.
15. 0.0060487.
16. 0.0030085.
17. 2.9538.
18. 0.9764.
19. 1.09325.
20. 18.175.
21. 0.00000000682.
22. 4.018.
23. 1.6727.
24. 4.537.
25. 1.5608.
26. 0.68114.
27. 4.116.
28. 0.001205.
29. 132.848.
30. 3025.71.
31. 428.
32. 5.00375.
33. 5.51.
34. 0.05639.
35. 0.6488.
36. 1.602.
37. 0.71016.
38. 0.063485.
39. 229.33 in.
40. 169.24 sq. in.
41. 19156.9 cu. in.
42. 5.411 ft.
43. 2.455 ft.
44. 22.805 in.
45. 103.92 sq. in.
46. 3827.5 sq. ft.
47. 0.5176.
48. 0.2610.
49. 67,500,000 mi.
50. 7.363.
51. 5.305 sq. ft.
52. $7590.
53. $7238.33.
54. $269.13.

Page 244.

1. $x=4$.
2. $x=2$.
3. $x=3$.
4. $x=2.47-$.
5. $x=1.55-$.
6. $x=8.63+$.
7. $x=2.16+$.
8. $x=3.39-$.
9. $x=3$.
10. $x=13$.
11. $x=5$.
12. $x=7.84+$.
13. 17.7+ yr.
14. 23.5+ yr.
15. 17.5 yr.
16. 20 yr.

Pages 244, 245.

4. 14.53.
5. 85.68.
6. $x=2.24-$.
7. $x=1.63+$.
8. $x=1$.
9. $x=\frac{1}{2}$.
10. $8937.50.
11. $1897.39.
12. $2120.

Pages 245, 251.

1. $(9n-a^2+a)(9n-a^2-a)$.
2. $(3a+b)(a-3b)$.
3. $(2x+2a-t)(2x-2a+t)$.
4. $(N^2+3N+9)(N^2-3N+9)$.
5. $V(V^2-5)(V^2-6)$.
6. $2(2k+1)(k+1)$.
7. $2(5R+6r)(3R+2r)$.
8. $(p^2+3pq-q^2)(p^2-3pq-q^2)$.
9. $(4N+1)(3N-2)$.
10. $(v^2+2)(2v^2-7)$.
11. $(x^4+1)(x^2+1)(x+1)(x-1)$.
12. $ab(a^2+b^2)(a+b)(a-b)$.
13. $(V^8+T^8)(V^4+T^4)(V^2+T^2)(V+T)(V-T)$.
14. $(m^4+n^4)(m^8-m^4n^4+n^8)$.
15. $(1+2a^3)(1-2a^3+4a^6)$.
16. $(a+1)(a-b-1)$.
17. $(t+1)(t-1)^2$.
18. $(y+2)(y-2)(3y+1)(3y-1)$.
19. $(n+1)(n-2)(n^4-2n^3+3n^2-2n+4)$.
20. $(H-1)(H+1)^2$.
21. $(v-1)(v-2)(v-4)$.
22. $(a+1)(a-1)(b+1)(b-1)$.
23. $(a+b+c)(a+b-c)(a-b+c)(a-b-c)$.
24. $(v^2+v-6w)(v^2-v+6w)$.
25. $(R+N)(R^2-RN+N^2+1)$.
26. $(h+t)(h-t-1)$.
27. $(n-1)(n-2)(n-3)$.
28. $(A+5)(A^2-5A-2)$.
29. $(m-2n^2)(m^4+2m^3n^2+4m^2n^4+8mn^6+16n^8)$.
30. $x(x-1)(x^3-x-1)$.
31. $(e^n+1)(e^{2n}+1)$.
32. $(G+a)(3G+b)$.
33. $(a+b)(a+c)(b+c)$.
34. $(5n-2+2a+3b)(5n-2-2a-3b)$.
35. $(6x^{n+3}-y^{n-1})(x^{n+3}-4y^{n-1})$.

36. $(x+y)(x-z)(y+z)(y-z-x)$.

37. $(n+a)(n+b+c)$.

40. $-3n^3+10n^2+2n+1$.

41. $3a-1$.

42. $a+2b-5c$.

43. $r^{3n}-r^{n+\frac{2}{n}}-r^n-r^{\frac{1}{n}}-r^{2n+\frac{1}{n}}+r^{\frac{2}{n}}$.

44. $t+2$.

45. $175m^3n^3(x-m)^2(x+m)^2(x+2m)$.

46. $x+y$; $(x^2+y^2)(x^3+y^3)(x^3-y^3)$.

47. Yes; No; Use rem. theorem.

48. $(p+1)(p-5)(2p+3)(3p^2+2)$.

49. $n=\dfrac{4a-4}{2a-1}$.

50. $3-n-5n^2+2n^3$.

51. $1-y^2+2y^3-y^4$.

52. $x^{-2}-x^{-1}y^{\frac{1}{2}}+y$.

53. $\dfrac{n+2}{n+1}$.

54. $\dfrac{2k^2-2}{4k^2+4k+1}$.

55. $\dfrac{r^4+r^2-2r+1}{r^7-r}$.

56. $(x-y)(y-z)(z-x)$.

38. $(a+x+b+c)(a+x-b-c)$.

39. $(m+n)(t+a)(t+b)(t-b)$.

57. $\dfrac{mn(n-m)(m^2-n^2)}{(m+n)(m^2+n^2)(m^2+mn+n^2)}$.

58. $\dfrac{1-x}{y}$.

59. $\dfrac{m}{1-n}$.

60. 1.

61. $\dfrac{m^4+6m^2n^2+n^4}{4m^3n+4mn^3}$.

62. 1.

64. $n=2\frac{11}{16}$.

65. $n=\frac{1}{12}$.

66. $x=1.239-$, $0.162-$.

67. $x=0.910-$.

68. $x^2-6x+4=0$.

69. Imaginary.

70. Impossible.

71. $n=5$.

72. $r=7$.

73. $v=2, t=5$; $v=-2, t=-5$; $v=\frac{9}{4}\sqrt{2}, t=-\frac{1}{2}\sqrt{2}$; $v=-\frac{9}{4}\sqrt{2}, t=\frac{1}{2}\sqrt{2}$.

74. $x=5, y=3$; $x=-3, y=-5$; $x=3, y=5$; $x=-5, y=-3$.

75. $a=-3$, $b=3$, $c=\frac{1}{2}$.

76. $n=\dfrac{d-2a\pm\sqrt{d^2-4ad+4a^2+8dS}}{2d}$.

77. $\sqrt{2}-1$.

78. $4-\sqrt{15}$.

79. $\dfrac{a-\sqrt{ab}+b}{a^2+ab+b^2}$.

80. 0.726.

81. $\frac{1}{2}\sqrt[4]{288}$.

82. $a^{\frac{5}{2}}c^{\frac{5}{6}}$.

83. $\frac{1}{64}m^{-\frac{1}{2}}n$.

84. $a^{-\frac{1}{6}}b^{-\frac{5}{4}}$.

85. $x^{15}y^{18}z^{-3}$.

86. $\dfrac{a}{bc^2}\sqrt[6]{b^4c^5}$.

87. $x=\sqrt{3}$.

92. 154.

93. 6, 7, 8, 9, 10.

94. $a, a+\frac{b-a}{n+1}, a+\frac{2b-2a}{n+1}, \ldots\ldots b.$

95. $-1860.$

96. 18th.

97. 24.

98. $34\frac{11}{64}.$

99. 6, 18, 54, 162, 486.

100. $a, a\sqrt[4]{\frac{b}{a}}, a\sqrt{\frac{b}{a}}, a\sqrt[4]{\frac{b^3}{a^3}}, b.$

101. $42+21\sqrt{2}.$

102. 154.9681956 cu. in.

103. $6\frac{5}{11}.$

104. 784 ft.

105. $\frac{1}{3}.$

106. $-\frac{117572a^5b^5\sqrt{ab}}{59049}.$

107. $4.946+.$

108. $1+\frac{1}{2}n^{-\frac{1}{2}}+\frac{3}{8}n^{-1}+\frac{5}{16}n^{-\frac{3}{2}}+\frac{35}{128}n^{-2}+\ldots\ldots$

109. $a^{2l}b^{2m}c^{2n}.$

110. $-2;\ \frac{1}{3};\ -1.$

111. $\sqrt[mn]{a^{my-nx}}.$

112. $a=1.$

113. $4m^{\frac{2}{3}}+n^{\frac{2}{3}}.$

114. $2n^2-8m.$

116. $x=15.05+.$

117. $\overline{1}.9112.$

118. $E=0.5029V^2D;\ D=\frac{1.9881E}{V^2}.$

119. $C'=\frac{EC}{E+Cr}.$

120. $h=\frac{R(\sqrt{g}-\sqrt{g'})}{\sqrt{g'}}.$

121. $u=\frac{t^2g+2s}{2t}.$

122. $v=7.64+.$

123. $r=\pm\frac{y}{\sqrt{4\pi y-4\pi^2R^2}};\ y=0.032+.$